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Parametric analysis of analytical solutions to one- and two-dimensional problems in couple-stress theory of elasticity

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In this paper the fundamentals of the asymmetric elasticity theory are used to consider one- and two-dimensional boundary-value problems: shear deformation of elastic infinite plane layer(plate); torsion of a ring rigidly fixed at the external contour due to rotation of the inner one; deformation of a plane washer caused by a rigid displacement of the internal contour relative to the external one; the Kirsch problem on unilateral extension of a plate loosened by a circular hole.

The solution of each problem is compared with the corresponding solution obtained in the framework of the classical theory of elasticity. The comparison is made in terms of macro-parameters introduced to characterize the degree of difference between these solutions. The analysis of the obtained results show that for each problem under consideration this difference is not essential. It is worthy of note that the macro-parameters used for comparison can be constructively measured by experiment. The obtained results can be used to outline a key diagram of experiments enabling one to detect the effects of "couple" response of the examined medium.

1 Introduction

Problems of material deformation, in which deformation of a medium is described not only by the displacement vector $\vec{u}$ but also by the rotation vector $\vec{\omega}$, have long been in the focus of scientists' interest. A medium simulated in such a way is commonly referred to as the Cosserat continuum and the theory describing its behavior is generally known in the literature as the couple-stress, asymmetric, or microstructure theory of elasticity.

In the framework of the Cosserat continuum theory [1]–[3] the displacements of particles in the examined medium are described in terms of two variables – an ordinary displacement field $\vec{u}$ and kinematically independent vector field $\vec{\omega}$, which is introduced to characterize small rotations of particles. Thus, in the couple-stress theory there are two independent kinematic unknown quantities, and the stress tensor $\boldsymbol{\sigma}$ and the couple-stress tensor $\boldsymbol{\mu}$ are asymmetric.

In the context of this theory the elastic behavior of isotropic linear medium is described by six elastic constants [3]–[5]: two Lamé constants and four new constants describing microstructure. In the case of quadratic-nonlinear medium the number of new constants increases to nine [6].

The history of microstructure theories goes back to works by W. Voigt [7], who was the first to introduce a model of the medium with rotational interaction of its particles for studying elastic properties of a crystal. An early effort to develop an elasticity theory with asymmetric stress tensor evidently belongs to E. Cosserat and F. Cosserat [8]. According to the Cosserat brothers' conception, which takes into account rotational interactions of material particles the most effective approach to the problems of stress-strain state in deformable solids is to introduce in the problem formulation the couple-stresses (moment of force per unit of area) in addition to the ordinary stresses (force per unit of area).

There has been a number of works reported in the literature in which the asymmetric theory is extended to the case of thermoelasticity and large deformations. Few works presenting solutions to a number of dynamic problems are also available in the literature. This is, for example, a systematic development of the modern theory by V. I. Erofeev [6], who considers the problem of propagation and interaction of elastic waves in solids with microstructure. Moreover, the idea of allowing for the internal rotation vector is often used for modeling plastic deformation in materials [9, 10]. However, a detailed discussion of these problems is beyond the scope of this paper, which is restricted to a static state of plane bodies in the framework of the elastic Cosserat continuum theory.

One of the means for taking into account couple stresses is the model of the pseudo-Cosserat continuum [11]–[19], which is based on the assumption that the displacement vector $\vec{u}$ of the medium points is related to the vectors of small rotations $\vec{\omega}$ by the equation

$$\vec{\omega} = \frac{1}{2}\,\mathrm{rot}\,\vec{u}\,. \tag{1}$$

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Thus, to describe the pseudo-Cosserat continuum in addition to available independent kinematic unknown (the displacement vector $\vec{u}$) one should introduce two additional variables - the asymmetric tensors of stresses $\boldsymbol{\sigma}$ and couple stresses $\boldsymbol{\mu}$. Here it should be noted that the asymmetric part of the stress and the symmetric part of couple stress cannot be derived directly from the physical equations. This was the reason why A. C. Eringen [20] named the theory of the pseudo-Cosserat continuum the theory of undefined couple stresses.

This version of the asymmetric theory suffers from incompleteness, since the number of physical constants for isotropic elastic body is reduced to 4. In this formulation the most-used quantities are the Young's modulus E, Poisson's ratio γ, the constant having the dimensions of length l, and dimensionless constant termed the bending modulus B [13, 17, 19, 21].

W. Nowacki [3] showed that the obtained structure of the equations for the pseudo-Cosserat continuum is such that if, in particular, the surface of elastic body is under the prescribed displacements, it is difficult, if ever possible, to specify arbitrarily the normal component of the rotation vector. Despite these drawbacks the pseudo-Cosserat theory has been well defined. Based on this theory a number of general theorems and integration schemes have been proposed and solutions to some problems have been developed [21]–[24].

The asymmetric theory of elasticity for the Cosserat continuum (especially for the pseudo-Cosserat continuum) was successfully used by many authors to construct exact analytical solutions. In the majority works the obtained solutions are analyzed and compared with the corresponding solutions of the classical elasticity theory. In this comparison, new physical constants specifying the contribution of the couple-stress components generally assume the values from the energetically admissible range. This can be explained by deficiency of information on the material constants of microstructure media, which is one of the main factors restricting further investigation of asymmetric media models.

There are also a few works in the literature dealing with identification of physical constants for the Cosserat and pseudo-Cosserat continuum. Measurements of elastic constants in static experiments are described in work [25]. More precise dynamic experiments (in particular, ultrasound) were used for identification of the Leru and pseudo-Cosserat models [26], the linear Cosserat continuum [27]–[30], and nonlinear Cosserat continuum (mixture) [31].

In some works (for example, [13] and [5]) a comparison between the solutions of the asymmetric and classical theories is carried out based on the analysis of the stress concentration coefficient and its dependence on the characteristic dimension of the stress concentrator. The analysis clearly demonstrated that compared to the classical theory the coefficient of the stress concentration increases (or decreases) with characteristic dimension of the concentrator. Although this fact is of obvious interest, the use of the concentration coefficient as a measurable parameter seems to be rather problematic. Thus, for example, an attempt to measure variation of the concentration index by the photoelasticity method has failed, since the resolving power of this method is too low to apply strictly to the desired characteristic dimension of the concentrator [32].

Considerable efforts have been spent on the development of analytical solutions to bending [12] and torsion [33] problems for rods with different cross-sections in terms of the asymmetric elasticity theory. In these studies a comparison of the couple-stress and classical solutions was based on the analysis of dependence of flexural and torsion stiffness on the characteristic dimension. Indeed, in the sense of experimental realization, stiffness is a well-measured parameter. However it is unlikely that the experimental measurements of flexural (torsion) stiffness can demonstrate the couple-stress response of the medium. This is due to the fact that such problems are missing one of the necessary conditions for the medium to show the couple-stress effect, namely a high stress gradient. The experiments reported in [32] support this statement.

Therefore the above approaches if viewed as the examples of the couple-stress response of the medium are thought to be unrealiable from the viewpoint of their experimental implementation.

The objectives of the present paper are as follows:

1. to develop and analyze exact analytical solutions to a number of one-dimensional and two-dimensional static boundary-value problems in the framework of elastic Cosserat theory;
2. to identify, based on the obtained solutions, measurable macro-quantities carrying information on "the couple" response of the examined material;
3. to determine and compare the degree of difference between the introduced macro-quantities for the Cosserat, pseudo-Cosserat, and classical continua;
4. to select problems that are most informative from the viewpoint of couple response effects, all other physical parameters being equal.

In this work, we develop the exact analytical solutions to four plane static problems in terms of the theory of elastic linear isotropic Cosserat continuum:

1. Shear deformation of a plane infinite layer (plate) fixed at both edges under the action of gravitational force. Despite its simplicity this problem is used in this work as an example of the stress-strain state at a low stress gradient at which the medium is expected to exhibit couple stress properties only slightly.
2. Torsion deformation of a ring rigidly fixed at the external contour due to rotation of its internal contour by a prescribed angle. Like the following problems, this case is characterized by a significant stress gradient compared to the first problem.
3. Deformation of a ring rigidly fixed at the external contour due to displacement of the internal contour by a prescribed value.

4. The Kirsch problem on uniaxial tension of an infinite plate loosened in the center by a circular hole. The extension of this problem to the pseudo-Cosserat continuum can be found in the work of R. D. Mindlin [13]. V. A. Palmov [5] determined the stress concentration in the vicinity of the circular hole in the context of the Cosserat continuum. It should be noted that the solution presented in [5] does not allow us to analyze in full measure the stress-strain state in the vicinity of the hole, in particular, to estimate the degree of hole distortion under deformation.

The solutions to the above problems are exact and are represented in dimensionless form using the Bessel function of various orders.

The analysis of the obtained solutions enabled us to determine the corresponding macro-quantities carrying information on the couple-stress response of the material. It is worthy of note that all these macro-quantities can be measured by experiment.

The extent of response of the introduced macro-quantities on the couple-stress behavior of the material is analyzed relative to the characteristic geometrical parameter of the problem.

2 Basic relations and problem formulations

As indicated above, in this paper we consider an elastic Cosserat continuum, which is described in [3] and [1], by the equilibrium equations

$$\vec{\nabla}\cdot\boldsymbol{\sigma}+\vec{X}=\vec{0}\,,\quad \boldsymbol{\sigma}^T:\vec{\mathbf{E}}+\vec{\nabla}\cdot\boldsymbol{\mu}+\vec{Y}=\vec{0}\,; \tag{2}$$

geometrical relations

$$\boldsymbol{\gamma}=\vec{\nabla}\vec{u}-\vec{\mathbf{E}}\cdot\vec{\omega}\,,\quad \boldsymbol{\chi}=\vec{\nabla}\vec{\omega}\,; \tag{3}$$

and physical equations

$$\boldsymbol{\sigma}=2\mu\boldsymbol{\gamma}^{(S)}+2\alpha\boldsymbol{\gamma}^{(A)}+\lambda I_1(\boldsymbol{\gamma})\mathbf{e}\,,\quad \boldsymbol{\mu}=2\gamma\boldsymbol{\chi}^{(S)}+2\varepsilon\boldsymbol{\chi}^{(A)}+\beta I_1(\boldsymbol{\chi})\mathbf{e}\,. \tag{4}$$

In terms of eqs. (2)–(4) the equilibrium equations for the displacement vector $\vec{u}$ and the rotation vector $\vec{\omega}$ can be written as

$$\begin{aligned}&(2\mu+\lambda)\,\mathrm{grad}\,\mathrm{div}\,\vec{u}-(\mu+\alpha)\,\mathrm{rot}\,\mathrm{rot}\,\vec{u}+2\alpha\,\mathrm{rot}\,\vec{\omega}+\vec{X}=\vec{0}\,,\\&(\beta+2\gamma)\,\mathrm{grad}\,\mathrm{div}\,\vec{\omega}-(\gamma+\varepsilon)\,\mathrm{rot}\,\mathrm{rot}\,\vec{\omega}+2\alpha\,\mathrm{rot}\,\vec{u}-4\alpha\vec{\omega}+\vec{Y}=\vec{0}\,.\end{aligned} \tag{5}$$

In (2)–(5), $\vec{\mathbf{E}}$ is the Levi-Cività tensor of third order; $(\cdot)^{(S)}$ is the symmetrization operation; $(\cdot)^{(A)}$ is the alternation operation of the tensor; $\vec{\nabla}(\cdot)$ is the nabla operator; $I_1(\cdot)$ is the first invariant of the tensor; $\vec{X}$ is the vector of mass forces; $\vec{Y}$ is the vector of mass couples; $\vec{u}$ is the displacement vector; $\vec{\omega}$ is the rotation vector; $\boldsymbol{\gamma}$ and $\boldsymbol{\chi}$ are asymmetric strain and torsion bending tensors; $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are asymmetric stress and couple stress tensors; μ,λ are the Lamé constants; $\alpha,\beta,\gamma,\varepsilon$ are physical constants of the material in the context of the Cosserat continuum.

In [25] and [39], the condition of specific positive internal energy is used to derive the following inequalities for material constants:

$$\begin{aligned}&3\lambda+2\mu+\alpha\geq 0\,,\quad 2\mu+\alpha\geq 0\,,\quad \alpha\geq 0\,,\\&3\beta+2\gamma\geq 0\,,\quad |\gamma-\varepsilon|\leq\gamma+\varepsilon\,,\quad \gamma+\varepsilon\geq 0\,.\end{aligned} \tag{6}$$

In what follows, we shall use three dimensionless quantities

$$A=l\sqrt{\frac{\alpha\mu}{(\alpha+\mu)(\gamma+\varepsilon)}}\,,\quad B=\frac{\alpha+\mu}{\alpha}\,,\quad C=\frac{\gamma-\varepsilon}{\gamma+\varepsilon}\,, \tag{7}$$

where l is the characteristic linear dimension for the problem under consideration.

From (6) it follows that $A>0,\ B\geq 1,\ |C|\leq 1$.

In this work, we also consider the pseudo-Cosserat continuum, which obeys relation (1).

The physical relations for the pseudo-Cosserat continuum are

$$\boldsymbol{\sigma}=2\mu\boldsymbol{\gamma}^{(A)}+\lambda I_1(\boldsymbol{\gamma})\mathbf{e}-\frac{1}{2}\vec{\nabla}\cdot\boldsymbol{\mu}\cdot\vec{\mathbf{E}}\,,\quad \boldsymbol{\mu}=2\gamma\boldsymbol{\chi}^{(S)}+2\varepsilon\boldsymbol{\chi}^{(A)}+\beta I_1(\boldsymbol{\chi})\mathbf{e}\,. \tag{8}$$

For the displacement vector $\vec{u}$ the equilibrium equation takes the form

$$\mu\nabla^2\,\vec{u}+(\mu+\lambda)\,\mathrm{grad}\,\mathrm{div}\,\vec{u}+\frac{1}{4}(\gamma+\varepsilon)\,\mathrm{rot}\,\mathrm{rot}\,\nabla^2\,\vec{u}+\vec{X}=\vec{0}\,. \tag{9}$$

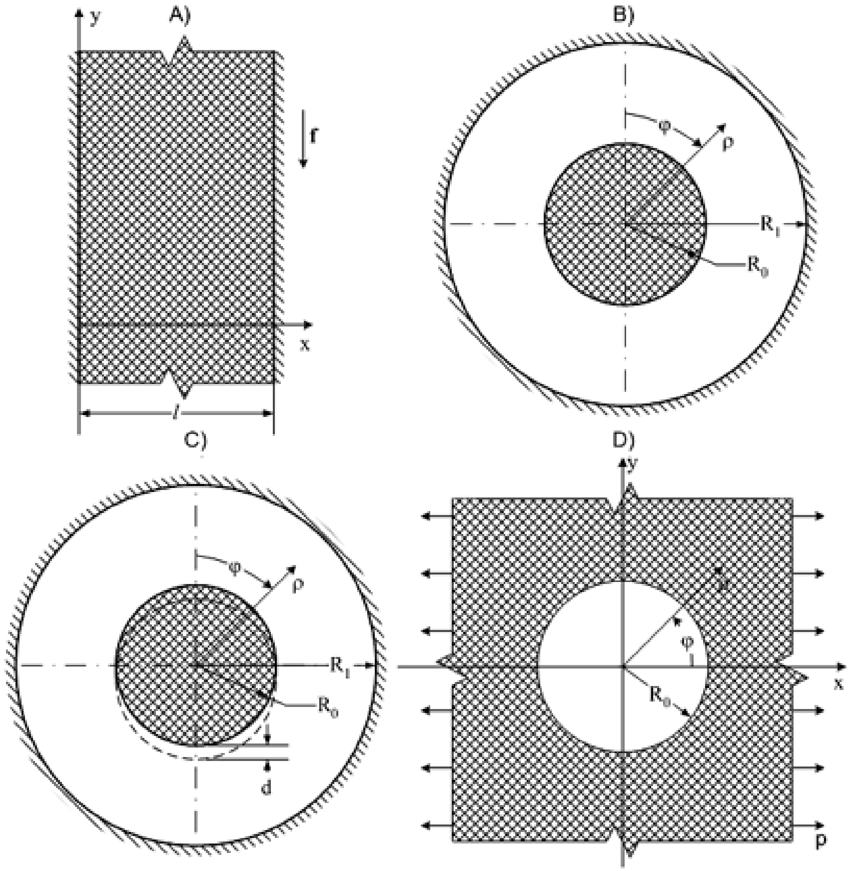

Fig. 1

The pseudo-Cosserat continuum is the corollary to the Cosserat continuum subject to the condition that $\alpha \to \infty$ [5]. Therefore in the following, the solutions corresponding to the pseudo-Cosserat continuum will be deduced from the Cosserat continuum solutions by applying the limiting transition.

The plane problems considered in this paper are schematically represented in Fig. 1:

1. A plane infinite layer (plate) of width l subjected to mass forces of intensity f, acting along the axis $0y$ is in the state of equilibrium. The left ($x = 0$) and the right ($x = l$) edges are fixed:

$$\vec{u}\big|_{x=0} = \vec{\omega}\big|_{x=0} = \vec{0}\,, \quad \vec{u}\big|_{x=l} = \vec{\omega}\big|_{x=l} = \vec{0}. \tag{10}$$

The solution in terms of the classical elasticity theory is given by [34]

$$u_y^*(x) = \frac{fx}{2}(x - l)\,. \tag{11}$$

2. A plane ring rigidly fixed at the external contour $\rho = R_1$ is under torsional stress due to rotation of the internal contour $\rho = R_0$ by the angle φ_0:

$$u_\varphi\big|_{\rho=R_0} = \varphi_0 \cdot R_0\,, \quad \omega_z\big|_{\rho=R_0} = \varphi_0\,, \quad u_\varphi\big|_{\rho=R_1} = 0\,, \quad \omega_z\big|_{\rho=R_1} = 0\,. \tag{12}$$

The solution in terms of the classical elasticity theory is written as [34]

$$u_\varphi^*(\rho) = \frac{R_0^2\varphi_0}{(1 - R_0^2)\rho} - \frac{R_0^2\varphi_0}{1 - R_0^2}\rho\,. \tag{13}$$

3. A plane ring rigidly fixed at the external contour $\rho = R_1$, is under shear deformation due to a rigid displacement of the internal contour $\rho = R_0$ by a magnitude d:

$$u_\rho\big|_{\rho=R_0} = -d\cos(\varphi)\,, \quad u_\varphi\big|_{\rho=R_0} = d\sin(\varphi)\,, \quad \omega_z\big|_{\rho=R_0} = 0\,, \quad u_\rho = u_\varphi = \omega_z\big|_{\rho=R_1} = 0\,. \tag{14}$$

The solution of the classical elasticity theory, known from the literature [34], is written as

$$\begin{aligned} u_\rho^*(\rho,\varphi) &= \left(C_1^* + \frac{C_2^*}{\rho^2} + C_3^*\rho^2 + C_4^*\ln(\rho)\right)\cos(\varphi)\,, \\ u_\varphi^*(\rho,\varphi) &= \left(-C_1^* + \frac{C_2^*}{\rho^2} + C_3^*\frac{3\lambda+5\mu}{\mu-\lambda}\rho^2 - C_4^*\left\{\ln(\rho) + \frac{\mu+\lambda}{3\mu+\lambda}\right\}\right)\sin(\varphi)\,. \end{aligned} \tag{15}$$

The constants $C_1^*, \ldots, C_4^*$ are defined from the corresponding boundary conditions (14).

4. An infinite plane loosened by a circular hole is under tensile stresses (the Kirsch problem). The periphery of the circular hole is free of the external stresses, and the plate at infinity is subjected to a tensile stress of constant intensity p in the direction of the axis Ox

$$\begin{aligned} &\sigma_{\rho\rho}\big|_{\rho=R_0} = 0\,, \quad \sigma_{\rho\varphi}\big|_{\rho=R_0} = 0\,, \quad \mu_{\rho z}\big|_{\rho=R_0} = 0\,, \\ &\sigma_{\rho\rho}\big|_{\rho\to\infty} = p_\rho\,, \quad \sigma_{\rho\varphi}\big|_{\rho\to\infty} = p_\varphi\,, \quad \mu_{\rho z}\big|_{\rho\to\infty} = 0\,. \end{aligned} \tag{16}$$

In the framework of classical elasticity theory this problem was first solved by G. Kirsch (1898), and later with slightly different approach by N. I. Muskhelishvili [35]:

$$\begin{aligned} u_\rho^*(\rho,\varphi) &= \left(C_1^*\rho + \frac{C_2^*}{\rho}\right) + \left(\frac{C_3^*}{\rho^3} + \frac{C_4^*}{\rho} + C_5^*\rho\right)\cos(2\varphi)\,, \\ u_\varphi^*(\rho,\varphi) &= \left(\frac{C_3^*}{\rho^3} - C_4^*\frac{\kappa-1}{(\kappa+1)\rho} - C_5^*\rho\right)\sin(2\varphi)\,, \end{aligned} \tag{17}$$

where

$$C_1^* = \frac{p(\kappa-1)}{8}\,, \quad C_2^* = \frac{pR_0^2}{4}\,, \quad C_3^* = -\frac{pR_0^4}{4}\,, \quad C_4^* = \frac{pR_0^2(\kappa+1)}{4}\,, \quad C_5^* = \frac{p}{4}\,, \quad C_6^* = 0\,.$$

The dimensionless quantity $\kappa = (3\mu+\lambda)/(\mu+\lambda)$.

3 Development of solutions for the Cosserat continuum

The methods and approaches used to solve the boundary-value problems in the framework of the asymmetric elasticity theory (for the Cosserat and pseudo-Cosserat continuum) are described in [36]–[39]. The authors of these works constructed the matrix of the fundamental solutions and volume, simple layer, and double layer potentials similar to those of Somigliana (see, e.g., [34]) in the classical theory of elasticity. By means of these potentials the solution of many boundary value problems of asymmetric elasticity theory was reduced to the corresponding integral equations [39]. The fundamental matrix in the classical theory of elasticity was actually discovered dy Lord Kelvin.

To find solutions to the boundary value problems (A)–(D) (Fig. 1) we use the approach of direct integration of the relative equilibrium equations.

The problems (B)–(D) (Fig. 1) are combined on the basis that they are most conveniently considered in the cylindrical coordinates (ρ, φ, z) in the form of truncated Fourier series:

$$\begin{aligned} &\vec{u}(\rho,\varphi) = \{u_\rho(\rho,\varphi), u_\varphi(\rho,\varphi), 0\}\,, \\ &\qquad u_\rho(\rho,\varphi) = U^{(0)}(\rho) + \sum_{n=1}^{N} U^{(n)}(\rho)\cos(n\varphi)\,, \\ &\qquad u_\varphi(\rho,\varphi) = V^{(0)}(\rho) + \sum_{n=1}^{N} V^{(n)}(\rho)\sin(n\varphi)\,, \\ &\vec{\omega}(\rho,\varphi) = \{0, 0, \omega_z(\rho,\varphi)\}\,, \\ &\qquad \omega_z(\rho,\varphi) = \omega^{(0)}(\rho) + \sum_{n=1}^{N} \omega^{(n)}(\rho)\sin(n\varphi)\,. \end{aligned} \tag{18}$$

Substituting relation (18) into eq. (5) we obtain a sequence of systems of ordinary differential equations for functions $U^{(n)}(\rho)$, $V^{(n)}(\rho)$, and $\omega^{(n)}(\rho)$, $n = 0, 1, ..., N$. Passing over the detailed construction of the general solutions for these functions [40], we write them for zero harmonic $n = 0$ as

$$
\begin{aligned}
U^{(0)}(\rho) &= C_1^{(0)}\rho + C_2^{(0)}\frac{1}{\rho}\,,\\
V^{(0)}(\rho) &= C_3^{(0)}\rho + \frac{C_4^{(0)}}{\rho} - C_5^{(0)}\frac{I_1(2A\rho)}{2A^2} - C_6^{(0)}\frac{K_1(2A\rho)}{2A^2}\,,\\
\omega^{(0)}(\rho) &= C_3^{(0)} - C_5^{(0)}\frac{BI_0(2A\rho)}{2A} + C_6^{(0)}\frac{BK_0(2A\rho)}{2A}\,,
\end{aligned}
\tag{19}
$$

for the first harmonic $n = 1$ as

$$
\begin{aligned}
U^{(1)}(\rho) &= C_1^{(1)} + \frac{C_2^{(1)}}{\rho^2} + C_3^{(1)}\rho^2 + C_4^{(1)}\ln(\rho) + C_5^{(1)}\frac{I_1(2A\rho)}{\rho} + C_6\frac{I_1(2A\rho)}{\rho}\,,\\
V^{(1)}(\rho) &= -C_1^{(1)} + \frac{C_2^{(1)}}{\rho^2} + C_3^{(1)}\frac{3\lambda+5\mu}{\mu-\lambda}\rho^2 - C_4^{(1)}\left\{\ln(\rho) + \frac{\mu+\lambda}{3\mu+\lambda}\right\}\\
&\quad + C_5^{(1)}\left\{\frac{I_1(2A\rho)}{\rho} - 2AI_0(2A\rho)\right\} + C_6^{(1)}\left\{\frac{K_1(2A\rho)}{\rho} + 2AK_0(2A\rho)\right\}\,,\\
w^{(1)}(\rho) &= C_3^{(1)}\frac{8\mu+4\lambda}{\mu-\lambda}\rho - C_4^{(1)}\frac{2\mu+\lambda}{(3\mu+\lambda)\rho} - 2A^2BI_1(2A\rho)C_5^{(1)} - 2A^2BK_1(2A\rho)C_6^{(1)},
\end{aligned}
\tag{20}
$$

and for the highest harmonics $n \geq 2$ as

$$
\begin{aligned}
U^{(n)}(\rho) &= \frac{C_1^{(n)}}{\rho^{(n+1)}} + \frac{C_2^{(n)}}{\rho^{(n-1)}} + C_3^{(n)}\rho^{(n-1)} + C_4^{(n)}\rho^{(n+1)} + \frac{C_5^{(n)}}{\rho}I_n(2A\rho) + \frac{C_6^{(n)}}{\rho}K_n(2A\rho)\,,\\
V^{(n)}(\rho) &= \frac{C_1^{(n)}}{\rho^{(n+1)}} + \frac{C_2^{(n)}}{\rho^{(n-1)}}\frac{n\lambda-2\lambda+n\mu-4\mu}{n\lambda+n\mu+2\mu} - C_3^{(n)}\rho^{(n-1)} - C_4^{(n)}\frac{n\lambda+2\lambda+n\mu+4\mu}{n\lambda+n\mu-2\mu}\rho^{(n+1)}\\
&\quad + C_5^{(n)}\left\{\frac{I_n(2A\rho)}{\rho} - \frac{2A}{n}I_{n-1}(2A\rho)\right\} + C_6^{(n)}\left\{\frac{K_n(2A\rho)}{\rho} + \frac{2A}{n}K_{n-1}(2A\rho)\right\}\,,\\
\omega^{(n)}(\rho) &= C_2^{(n)}\frac{2n\lambda-2\lambda+4n\mu-4\mu}{n\lambda+n\mu+2\mu}\frac{1}{\rho^n} - C_4^{(n)}\frac{2n\lambda+2\lambda+4n\mu+4\mu}{n\lambda+n\mu-2\mu}\rho^n\\
&\quad - C_5^{(n)}\frac{2A^2B}{n}I_n(2A\rho) - C_6^{(n)}\frac{2A^2B}{n}K_n(2A\rho)\,.
\end{aligned}
\tag{21}
$$

In relations (19)–(21), $I_n(\rho)$ is the modified Bessel function of the first kind in the limit tending to infinity at $\rho \to \infty$ and $K_n(\rho)$ is the modified Bessel function of the second kind or the MacDonald function in the limit tending to zero at $\rho \to \infty$, $C_1^{(n)}, \ldots C_6^{(n)}$ are the constants obtained from the boundary conditions.

Expressions (19)–(21) are also the solutions to eq. (9), which describes the boundary value problem for the pseudo-Cosserat continuum in the case $B = 1$ (see (7)) which corresponds to the case of $\alpha \to \infty$.

The problem on shear deformation of a layer (plate) has a rather simple solutions, which can be readily written as

$$
\begin{aligned}
u_y(x) &= C_1 + C_2x + C_3\,\mathrm{e}^{2Ax} + C_4\,\mathrm{e}^{-2Ax} + \frac{f}{2}x^2\,,\\
\omega_z(x) &= \frac{C_2}{2} + C_3AB\,\mathrm{e}^{2Ax} - C_4AB\,\mathrm{e}^{-2Ax} + \frac{f}{2}x\,,\\
\gamma_{xy}(x) &= \frac{C_2}{2} - C_3A(B-2)\,\mathrm{e}^{2Ax} + C_4A(B-2)\,\mathrm{e}^{-2Ax} + \frac{f}{2}x\,,\\
\gamma_{yx}(x) &= \frac{C_2}{2} + C_3AB\,\mathrm{e}^{2Ax} - C_4AB\,\mathrm{e}^{-2Ax} + \frac{f}{2}x\,,\\
\chi_{xz}(x) &= 2C_3A^2B\,\mathrm{e}^{2Ax} + 2C_4A^2B\,\mathrm{e}^{-2Ax} + \frac{f}{2}\,,\\
\sigma_{xy}(x) &= fx + C_2,\\
\sigma_{yx}(x) &= fx + C_2 + 4C_3A\,\mathrm{e}^{2Ax} - 4C_4A\,\mathrm{e}^{-2Ax}\,,\\
\mu_{xz}(x) &= 2C_3\,\mathrm{e}^{2Ax} + 2C_4\,\mathrm{e}^{-2Ax} + \frac{f}{2A^2B}\,,\\
\mu_{zx}(x) &= C\mu_{xz}(x)\,.
\end{aligned}
\tag{22}
$$

Here and in the following A, B, C are dimensionless complexes (7), and all quantities are nondimensional. The linear quantities are referred to characteristic dimensions of the problem l, the stresses σ_{ij} to the shear modulus μ, and the couple stresses μ_{ij} are multiplied by l/μ.

Boundary conditions (10) yield the system of linear algebraic equations, from which the constants $C_1, \ldots, C_4$ are defined:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1/2 & AB & -AB \\ 1 & 1 & \mathrm{e}^{2A} & \mathrm{e}^{-2A} \\ 0 & 1/2 & AB\,\mathrm{e}^{2A} & -AB\,\mathrm{e}^{-2A} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ -1/2 \end{bmatrix}.$$

The numerical simulation readily demonstrated that for a prospective experiment the maximum axial displacement $u_y(1/2)$ can be used as a measurable parameter of the problem. The degree of difference between the solution of the couple-stress theory (22) and that of the classical theory (11) is defined by the quantity

$$\delta_1 = \left| \frac{u_y(1/2) - u_y^*(1/2)}{u_y^*(1/2)} \right| \cdot 100\%\,. \tag{23}$$

Hereinafter, $(.)^*$ denotes the corresponding quantities relevant to the classical elasticity theory.

Solution to the problem of ring torsion is obtained under the requirement of retaining the first term in (18) and taking into account that $u_\rho(\rho) = 0$:

$$\begin{aligned} u_\varphi(\rho) &= C_1\rho + C_2\frac{1}{\rho} - C_3\frac{I_1(2A\rho)}{2A^2} - C_4\frac{K_1(2A\rho)}{2A^2}\,, \\ \omega_z(\rho) &= C_1 - C_3\frac{BI_0(2A\rho)}{2A} + C_4\frac{BK_0(2A\rho)}{2A}\,, \\ \sigma_{\rho\varphi}(\rho) &= -\frac{2C_2}{\rho^2} + C_3\frac{I_1(2A\rho)}{2A^2\rho} + C_4\frac{K_1(2A\rho)}{2A^2\rho}\,, \\ \sigma_{\varphi\rho}(\rho) &= -\frac{2C_2}{\rho^2} + C_3\left(\frac{I_1(2A\rho)}{2A^2\rho} - \frac{2I_0(2A\rho)}{A}\right) + C_4\left(\frac{K_1(2A\rho)}{2A^2\rho} + \frac{2K_0(2A\rho)}{A}\right), \\ \mu_{\rho z}(\rho) &= -C_3\frac{I_1(2A\rho)}{2A^2} - C_4\frac{K_1(2A\rho)}{2A^2}\,. \end{aligned} \tag{24}$$

Here the value of the external contour radius R_1 is taken as a characteristic linear dimension.

The constants $C_1, \ldots, C_4$ are evaluated from the boundary conditions (12) for particular values of dimensionless quantities A and B (see (7)) from the solutions to the following system of linear algebraic equations:

$$\begin{bmatrix} R_0 & \frac{1}{R_0} & -\frac{I_1(2AR_0)}{2A^2} & -\frac{K_1(2AR_0)}{2A^2} \\ 1 & 0 & -\frac{BI_0(2AR_0)}{2A} & \frac{BK_0(2AR_0)}{2A} \\ 1 & 1 & -\frac{I_1(2A)}{2A^2} & -\frac{K_1(2A)}{2A^2} \\ 1 & 0 & -\frac{BI_0(2A)}{2A} & \frac{BK_0(2A)}{2A} \end{bmatrix} \cdot \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} \varphi_0 R_0 \\ \varphi_0 \\ 0 \\ 0 \end{bmatrix}.$$

Based on the results of numerical simulation we found that in this problem the turning couple on the internal contour M can be used as experimentally defined parameter

$$M = \int_0^{2\pi} \sigma_{\rho\varphi}(R_0)R_0^2 d\varphi + \int_0^{2\pi} \mu_{\rho z}(R_0)R_0 d\varphi\,, \tag{25}$$

and the degree of difference between the couple-stress (24) and classical (13) solutions is defined by

$$\delta_2 = \left| \frac{M - M^*}{M^*} \right| \cdot 100\,\%. \tag{26}$$

Solution to the problem of ring deformation is obtained under the requirement of retaining the second term in (18) for $n = 1$ (see (20) below):

$$
\begin{aligned}
u_\rho(\rho,\varphi) &= \cos(\varphi)\left(C_1 + \frac{C_2}{\rho^2} + C_3\rho^2 + C_4\ln(\rho) + C_5\frac{I_1(2A\rho)}{\rho} + C_6\frac{K_1(2A\rho)}{\rho}\right),\\
u_\varphi(\rho,\varphi) &= \sin(\varphi)\Bigg(-C_1 + \frac{C_2}{\rho^2} + C_3\frac{3\lambda+5\mu}{\mu-\lambda}\rho^2 - C_4\left\{\ln(\rho) + \frac{\mu+\lambda}{3\mu+\lambda}\right\}\\
&\qquad + C_5\left\{\frac{I_1(2A\rho)}{\rho} - 2AI_0(2A\rho)\right\} + C_6\left\{\frac{K_1(2A\rho)}{\rho} + 2AK_0(2A\rho)\right\}\Bigg),\\
\omega_z(\rho,\varphi) &= \sin(\varphi)\left(C_3\frac{8\mu+4\lambda}{\mu-\lambda}\rho - \frac{C_4(2\mu+\lambda)}{(3\mu+\lambda)\rho} - 2A^2BI_1(2A\rho)C_5 - 2A^2BK_1(2A\rho)C_6\right),\\
\sigma_{\rho\rho}(\rho,\varphi) &= \cos(\varphi)\left(-C_2\frac{4}{\rho^3} + C_3\frac{4(\mu+\lambda)}{\mu-\lambda}\rho + C_4\frac{2(3\mu+2\lambda)}{(3\mu+\lambda)\rho} + C_5S^{(5)}_{\rho\rho}(\rho) + C_6S^{(6)}_{\rho\rho}(\rho)\right),\\
\sigma_{\rho\varphi}(\rho,\varphi) &= \sin(\varphi)\left(-C_2\frac{4}{\rho^3} + C_3\frac{4(\mu+\lambda)}{\mu-\lambda}\rho - C_4\frac{2\mu}{(3\mu+\lambda)\rho} + C_5S^{(5)}_{\rho\varphi}(\rho) + C_6S^{(6)}_{\rho\varphi}(\rho)\right),
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
S^{(5)}_{\rho\rho}(\rho) &= \frac{4A}{\rho}I_0(2A\rho) - \frac{4}{\rho^2}I_1(2A\rho), \quad S^{(6)}_{\rho\rho}(\rho) = -\frac{4A}{\rho}K_0(2A\rho) - \frac{4}{\rho^2}K_1(2A\rho),\\
S^{(5)}_{\rho\varphi}(\rho) &= \frac{4A}{\rho}I_0(2A\rho) - \frac{4}{\rho^2}I_1(2A\rho), \quad S^{(6)}_{\rho\varphi}(\rho) = -\frac{4A}{\rho}K_0(2A\rho) - \frac{4}{\rho^2}K_1(2A\rho).
\end{aligned}
$$

The radius of the external contour R_1 is taken as a characteristic linear dimension.

The constants $C_1, \ldots, C_6$ are evaluated from boundary conditions (14), written in the form of the system of linear algebraic equations:

$$
\begin{bmatrix}
1 & \frac{1}{R_0^2} & R_0^2 & \ln(R_0) & U_5(R_0) & U_6(R_0)\\
-1 & \frac{1}{R_0^2} & \frac{R_0^2(3\lambda+5\mu)}{\mu-\lambda} & -\frac{\mu+\lambda}{3\mu+\lambda} - \ln(R_0) & V_5(R_0)3 & V_6(R_0)\\
0 & 0 & \frac{R_0(4\lambda+8\mu)}{\mu-\lambda} & -\frac{2\mu+\lambda}{R_0(3\mu+\lambda)} & \omega_5(R_0) & \omega_6(R_0)\\
1 & 1 & 1 & 0 & U_5(1) & U_6(1)\\
-1 & 1 & \frac{3\lambda+5\mu}{\mu-\lambda} & -\frac{\mu+\lambda}{3\mu+\lambda} & V_5(1) & V_6(1)\\
0 & 0 & \frac{4\lambda+8\mu}{\mu-\lambda} & -\frac{2\mu+\lambda}{3\mu+\lambda} & \omega_5(1) & \omega_6(1)
\end{bmatrix}
\begin{bmatrix} C_1\\ C_2\\ C_3\\ C_4\\ C_5\\ C_6 \end{bmatrix}
=
\begin{bmatrix} d\\ d\\ 0\\ 0\\ 0\\ 0 \end{bmatrix}.
$$

The results of numerical simulation demonstrated that the degree of response of the internal contour F_y can be conveniently used as a measurable quantity

$$
F_y = \int_0^{2\pi} \left(\sigma_{\rho\varphi}(R_0,\varphi)\sin(\varphi) + \sigma_{\rho\rho}(R_0,\varphi)\cos(\varphi)\right) R_0 d\varphi, \tag{28}
$$

and the degree of difference between the couple-stress (27) and classical (15) solutions is defined by

$$
\delta_3 = \left|\frac{F_y - F_y^*}{F_y^*}\right| \cdot 100\%. \tag{29}
$$

Solution of the Kirsch problem is obtained under the requirement of retaining first and third terms in (18) for $n = 0$ (see (19)) and $n = 2$ (see (21)):

$$\begin{aligned} u_\rho(\rho,\varphi) &= C_1\rho + \frac{C_2}{\rho} + \left(\frac{C_3}{\rho^3} + \frac{C_4}{\rho} + C_5\rho + C_6\rho^3 + C_7U_7(\rho) + C_8U_8(\rho)\right)\cos(2\varphi)\,,\\ u_\varphi(\rho,\varphi) &= \left(\frac{C_3}{\rho^3} - C_4\frac{\kappa-1}{(\kappa+1)\rho} - C_5\rho - C_6\frac{\kappa+3}{\kappa-3}\rho^3 + C_7V_7(\rho) + C_8V_8(\rho)\right)\sin(2\varphi)\,,\\ \omega_z(\rho,\varphi) &= \left(\frac{C_4}{\rho^2} - C_6\frac{3(\kappa+1)}{3-\kappa}\rho^2 + C_7\omega_7(\rho) + C_8\omega_8(\rho)\right)\sin(2\varphi)\,. \end{aligned} \tag{30}$$

As a linear characteristic dimension we choose the radius of a circular hole R_0 in the center of an infinite plate.

The constants determined from the boundary conditions (16), are

$$C_1 = \frac{p(\kappa-1)}{8}\,,\quad C_2 = \frac{pR_0^2}{4}\,,$$

$$C_3 = -\frac{pR_0^4}{4}\cdot\left(\frac{2L(BL^2+4\kappa+4)K_0(L)}{L^2(2BLK_0(L)+(BL^2+4B+2\kappa+2)K_1(L))} + \frac{(BL^4+4BL^2+2L^2+2L^2\kappa+16\kappa+16)K_1(L)}{L^2(2BLK_0(L)+(BL^2+4B+2\kappa+2)K_1(L))}\right),$$

$$C_4 = \frac{pR_0^2(\kappa+1)}{4}\cdot\frac{B(2LK_0(L)+(4+L^2)K_1(L))}{2BLK_0(L)+(BL^2+4B+2\kappa+2)K_1(L)}\,,$$

$$C_5 = \frac{p}{4}\,,\quad C_6 = 0\,,\quad C_7 = 0\,,\quad C_8 = \frac{p(\kappa+1)L}{2(2BLK_0(L)+(BL^2+4B+2\kappa+2)K_1(L))}\,.$$

Here for the sake of shorthand writing we introduce the dimensionless quantity $L = 2AR_0$.

The numerical simulation demonstrated that the parameter D , characterizing the degree of distortion of the hole boundary is convenient to use as an experimentally measured parameter

$$D = \left|\frac{u_\rho(R_0,0)}{u_\rho(R_0,\pi/2)}\right|, \tag{31}$$

and the degree of difference between the couple-stress (30) and classical (17) solution is defined by

$$\delta_4 = \left|\frac{D-D^*}{D^*}\right|\cdot 100\%\,. \tag{32}$$

4 Analysis of solutions

In this section, we analyze the defined quantities δ_1 , δ_2 , δ_3 , δ_4 from (23), (26), (29), (32), respectively, for the solutions of all examined problems. As mentioned above these quantities characterize the degree of the material response to the couple-stress properties. It must be emphasized again that all parameters specifying the values of δ_i can be constructively measured in presumable experiments.

The physical parameters for numerical analysis are designated as [29]: $\lambda = 2.096\cdot 10^9\,\mathrm{N/m^2}$, $\mu = 1.033\cdot 10^9\,\mathrm{N/m^2}$, $\alpha = 1.148\cdot 10^8\,\mathrm{N/m^2}$, $\gamma = 4.1\cdot 10^6\,\mathrm{N}$, $\varepsilon = 1.312\cdot 10^5\,\mathrm{N}$. For construction of plots (B) and (C) we also use the quantity $R_0 = 0.1$, characterizing the dimensionless radius of the internal contour.

A key to the analysis is the equality of physical parameters in the four examined problems. This allows us to make a comparison between the examined problems and to estimate in such a way the informative value of each problem from the viewpoint of the medium response to couple stresses. In Fig. 2 the introduced relative parameters δ_1 , δ_2 , δ_3 , δ_4 (32) are plotted against the corresponding linear characteristic dimension of the problem.

Continuous lines correspond to the Cosserat continuum and dashed lines correspond to the pseudo Cosserat problem.

The dependence of all quantities characterizing the stres-strain state on the parameter α has been analyzed. The results of the analysis show that the Cosserat continuum solution tends to the classical solution as $\alpha \to 0$, and to the pseudo-Cosserat

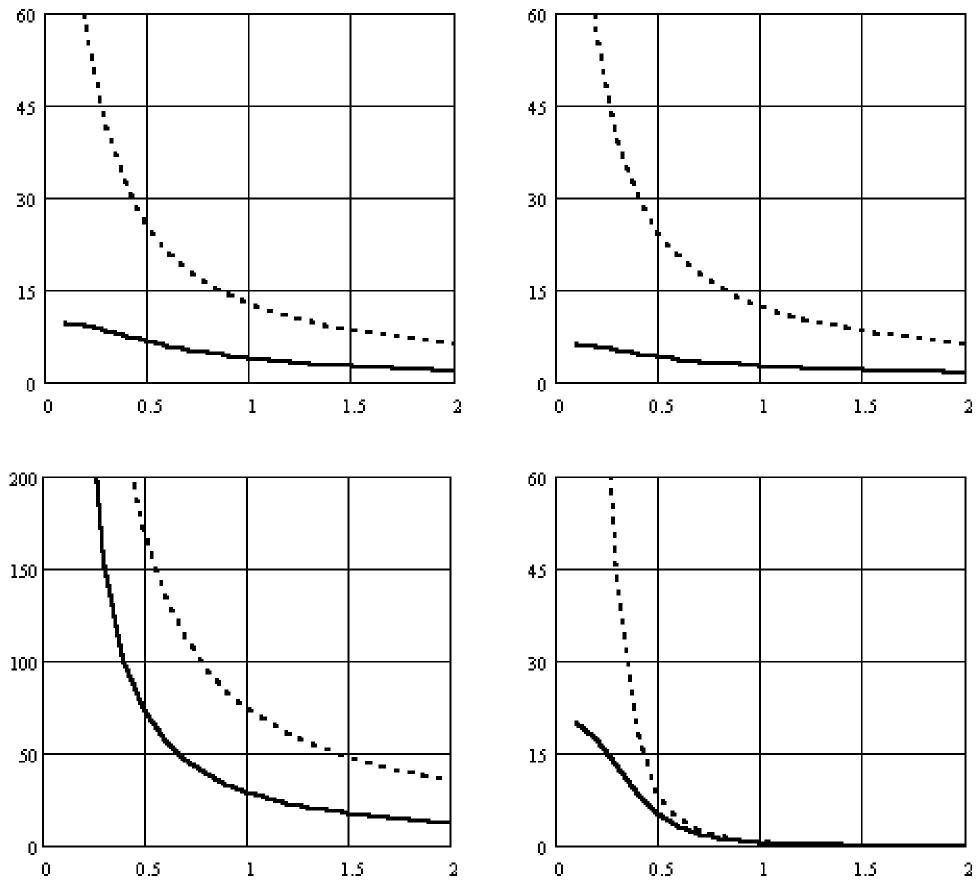

Fig. 2

solution as $\alpha \to \infty$. This accounts for the fact that in Fig. 2 the values of the relative parameters for the pseudo-Cosserat continuum are higher than the corresponding values for the Cosserat continuum.

As is evident from comparison of four plots in Fig. 2 the most informative is the problem of flat ring torsion (C) and the problem (D). As might be expected the problem of shear deformation (A) has proved to be less informative due to the absence of large stress gradients. The results of comparative analysis are used to trace out a line of future experimental studies which are supposed to reveal the effect of couple stress on the material behavior.

5 Conclusions

In this paper, the exact analytical solutions to four boundary-value problems of the asymmetric elasticity theory were presented. The numerical simulation was made to identify parameters, which can be measured in experiments intended for detecting couple effects in a medium. Graphic relationships were constructed to estimate the informative value of the introduced parameters from the viewpoint of their ability to reveal the effect of couple-stress on material properties. The obtained results allow us to outline the key diagrams of future experiments.

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