European Congress on Computational Methods in Applied Sciences and Engineering
ECCOMAS 2004
P. Neittaanmäki, T. Rossi, K. Majava, and O. Pironneau (eds.)
R. Owen and M. Mikkola (assoc. eds.)
Jyväskylä, 24–28 July 2004

ANALYTICAL AND NUMERICAL SOLUTIONS OF TWO-DIMENSIONAL PROBLEMS OF ASYMMETRIC ELASTICITY THEORY

Valery V. Korepanov***, Michail A. Kulesh*****, Valery P. Matveenko*****,**
Igor N. Shardakov*

*Ural Branch of RAS
Academic Korolev street, 1, 614013, Institute of Continua Media Mechanics, Perm, Russia
e-mails: kvv@icmm.ru, kma@icmm.ru, mvp@icmm.ru, shardakov@icmm.ru

Key words: Asymmetric elasticity theory, finite element.

Abstract. *In this work new analytical solutions to the following problems of asymmetric elasticity theory are presented: torsion and deformation of a rigid ring in elasticity region, unilateral extension of a plate with a circular hole, shear deformation of elastic infinite plane layer.*

To enlarge the number of the aforementioned problems the finite element method algorithm has been developed for solving two-dimensional problems of asymmetric elasticity theory. Numerical solutions are found for several problems in which couple effects manifest themselves more "dramatic" than in the problems with realized analytical solutions, for example, the problem of unilateral extension of a plate with several circular holes.

The algorithm of sensitivity analysis method is proposed to use for estimate an influence of mechanical characteristic values on stress-strain state.

The analysis of several problems has been carried out on the basis of the obtained analytical solutions by the finite element algorithm applying sensitivity analysis techniques with the purpose to reveal the cases where the couple behavior of elastic media is more pronounced. Summarizing theoretical results we propose experimental procedures for identification of material constants of the asymmetric elasticity theory.

1 INTRODUCTION

In the framework of the Cosserat continuum theory [1]-[3] the displacements of particles in the examined medium are described in terms of two variables - an ordinary displacement field $\vec{u}$ and kinematically independent vector field $\vec{\omega}$, which is introduced to characterize small rotations of particles. Thus, the couple-stress theory operates with two independent kinematic unknown quantities, and the stress tensor $\tilde{\sigma}$ and the couple-stress tensor $\tilde{\mu}$ are asymmetric.

In this theory, the elastic behavior of isotropic linear medium is characterized by six elastic constants [3]-[5]: two Lamé constants and four new constants describing microstructure.

One of the ways to take into account the couple-stresses is to use the model of pseudo-Cosserat continuum [6]-[11], which is based on the assumption that the displacement vector $\vec{u}$ of the medium points is related to the vectors of small rotations $\vec{\omega}$ by the equation

$$\vec{\omega} = \frac{1}{2}\, rot\, \vec{u}. \tag{1}$$

Thus, for consideration of pseudo-Cosserat continuum we have one independent kinematic unknown (the displacement vector $\vec{u}$) and we should introduce two additional variables – the asymmetric tensors of stresses $\tilde{\sigma}$ and couple stresses $\tilde{\mu}$. However, the asymmetric part of the stress and the symmetric part of the couple stress cannot be derived directly from the physical equations, which was the reason why Eringen [12] named the theory of the pseudo-Cosserat continuum the theory of undefined couple-stresses.

In this version of the asymmetric theory the number of physical constants for isotropic elastic body is reduced to 4. For example, the most-used quantities are the Young's modulus E, Poisson's ratio γ, the constant having the dimensions of length l, and dimensionless constant termed the bending modulus B [8], [9], [11], [13].

Nowacki [3] showed that the obtained structure of the equations for pseudo-Cosserat continuum is such that if, in particular, the surface of elastic body is under the prescribed displacements, it is difficult, if ever possible, to set arbitrarily the normal component of the rotation vector. Despite these drawbacks the pseudo-Cosserat theory has been well elaborated. Within this theory a number of general theorems and integration schemes have been proposed and solutions to some problems have been developed [13]-[16].

The asymmetric theory of elasticity has been successfully used for constructing a number of exact analytical solutions for the Cosserat continuum. The obtained solutions are analyzed and compared with the corresponding solutions of the classical elasticity theory. In the proposed treatments, the values of new physical constants specifying the contribution of the couple-stress components are generally assigned from their energetically admissible range. This is explained by deficiency of information about the material constants of microstructure media. In the asymmetric elasticity theory the problem with identification of material constants is accentuated by a shortage of experiments, which could fix the fact of couple-stress response of a material. These circumstances restrict practical application of the asymmetric elasticity theory.

There are only few works in the literature dealing with identification of physical constants for the Cosserat and pseudo-Cosserat continuum. In work [17], the elastic constants were defined using the results of static experiments. More precise dynamic experiments (in particular, ultrasound) were used for identification of the Leru and pseudo-Cosserat models [18], and for identification of the linear Cosserat continuum [19]-[22].

In works [8] and [5] a comparison of asymmetric solutions with those of classical theories is based on the analysis of the stress concentration coefficient and its dependence on the characteristic dimension of the stress concentrator. The analysis clearly demonstrated that compared to the classical theory the coefficient of the stress concentration increases with decrease of the characteristic dimension of the concentrator. This fact is of considerable importance, although the use of the concentration coefficient as a measurable parameter seems to be rather problematic. Thus, for example, an attempt to measure variation of the concentration factor by the method of photo-elasticity has failed, since the resolving power of this method is too low for the desired characteristic dimension of the concentrator [23].

In several studies the asymmetric elasticity theory was used to obtain analytical solutions to bending [7] and torsion [24] problems for rods with different cross-sections. In these papers a comparison of the couple-stress and classical solutions was based on the analysis of dependence of flexural and torsion stiffness on the characteristic dimension. The experimental measurements of flexural (torsion) stiffness did not reveal couple-stress response of the medium. This is due to the fact that in these problems the necessary conditions for the couple-stress effect, namely, high stress gradients are missing. The experiments reported in [23] support this statement.

The motivation of this work is to find solutions to the problems providing new possibilities for identification of material constants of the asymmetric elasticity theory and realization of experiments revealing couple-stress response of the material.

The objectives of the present paper are as follows: to develop and analyze analytical and numerical solutions to a number of one-dimensional and two-dimensional static boundary-value problems in the framework of elastic Cosserat theory; to identify, based on the obtained solutions, measurable macro-quantities carrying information on "the couple-stress" response of the examined material; to determine and compare the degree of difference between the introduced macro-quantities for the Cosserat, and classical continua; to select problems that are most informative from the viewpoint of couple-stress effects using techniques of the sensitivity analysis.

In this work, we develop the solutions to a number of plane static problems in terms of the theory of elastic linear isotropic Cosserat continuum: shear deformation of a plane infinite layer (plate) fixed at both edges under the action of gravitational force; torsion deformation of a ring rigidly fixed at the external contour due to rotation of its internal contour by a prescribed angle; deformation of a ring rigidly fixed at the external contour due to displacement of the internal contour by a prescribed value; the problem on uniaxial tension of an infinite plate weakened in the center by a circular hole; extension of a plate weakened by several circular holes; extension of a plate with a crack in the center.

2 BASIC RELATIONS OF ASYMMETRIC ELASTICITY THEORY

Let us consider the basic relations of the elastic Cosserat continuum [3], which are used for development of analytical solutions.

the equilibrium equations

$$\vec{\nabla} \cdot \tilde{\sigma} + \vec{X} = \vec{0}, \quad \tilde{\sigma}^T : \vec{\mathbf{E}} + \vec{\nabla} \cdot \tilde{\mu} + \vec{Y} = \vec{0}; \tag{2}$$

geometrical relations

$$\tilde{\gamma} = \vec{\nabla}\vec{u} - \vec{\mathbf{E}} \cdot \vec{\omega}, \quad \tilde{\chi} = \vec{\nabla}\vec{\omega}; \tag{3}$$

and physical equations

$$\tilde{\sigma} = 2\mu\tilde{\gamma}^{(S)} + 2\alpha\tilde{\gamma}^{(A)} + \lambda I_1(\tilde{\gamma})\mathbf{e}, \quad \tilde{\mu} = 2\gamma\tilde{\chi}^{(S)} + 2\varepsilon\tilde{\chi}^{(A)} + \beta I_1(\tilde{\chi})\mathbf{e}. \tag{4}$$

In terms of equations (2)-(4) the equilibrium equations for the displacement vector $\vec{u}$ and the rotation vector $\vec{\omega}$ can be written as:

$$\begin{aligned} &(2\mu + \lambda) grad\, div\, \vec{u} - (\mu + \alpha) rot\, rot\, \vec{u} + 2\alpha\, rot\, \vec{\omega} + \vec{X} = \vec{0}, \\ &(\beta + 2\gamma) grad\, div\, \vec{\omega} - (\gamma + \varepsilon) rot\, rot\, \vec{\omega} + 2\alpha\, rot\, \vec{u} - 4\alpha\vec{\omega} + \vec{Y} = \vec{0}. \end{aligned} \tag{5}$$

In (2)-(5), $\vec{\mathbf{E}}$ is the Levi-Cività tensor of third order; $(\cdot)^{(S)}$ is the symmetrization operation; $(\cdot)^{(A)}$ is the alternation operation of the tensor; $\vec{\nabla}(\cdot)$ is the nabla operator; $I_1(\cdot)$ is the first invariant of the tensor; $\vec{X}$ is the vector of mass forces; $\vec{Y}$ is the vector of mass moments; $\vec{u}$ is the displacement vector; $\vec{\omega}$ is the rotation vector; $\tilde{\gamma}$ and $\tilde{\chi}$ are the asymmetric strain and torsion-bending tensors; $\tilde{\sigma}$ and $\tilde{\mu}$ are asymmetric stress and couple-stress tensors; μ, λ are the Lamé constants; $\alpha, \beta, \gamma, \varepsilon$ are physical constants of the material in the context of the Cosserat continuum.

In [17] and [25], the condition of positive specific internal energy is used to derive the following inequalities for material constants:

$$\begin{aligned} &3\lambda + 2\mu + \alpha \geq 0, \quad 2\mu + \alpha \geq 0, \quad \alpha \geq 0, \\ &3\beta + 2\gamma \geq 0, \quad |\gamma - \varepsilon| \leq \gamma + \varepsilon, \quad \gamma + \varepsilon \geq 0. \end{aligned} \tag{6}$$

In what follows, we shall use three dimensionless quantities:

$$A = l\sqrt{\frac{\alpha\mu}{(\alpha + \mu)(\gamma + \varepsilon)}}, \quad B = \frac{\alpha + \mu}{\alpha}, \quad C = \frac{\gamma - \varepsilon}{\gamma + \varepsilon}. \tag{7}$$

where l is the characteristic linear dimension for the problem under consideration.

From (6) it follows that $A > 0$, $B \geq 1$, $|C| \leq 1$.

For numerical realization by the finite element method we use the variational equation, which in the context of the couple-stress theory of elasticity is written as:

$$\int\limits_V (\tilde{\sigma} : \delta\tilde{\gamma} + \tilde{\mu} : \delta\tilde{\chi})\, dV - \int\limits_V \left(\vec{X} \cdot \delta\vec{u} + \vec{Y} \cdot \delta\vec{\omega}\right) dV = \int\limits_S (\vec{p} \cdot \delta\vec{u} + \vec{m} \cdot \delta\vec{\omega})\, dS \qquad (8)$$

where $\vec{p}$ is the basic vector, $\vec{m}$ is the basic moment.

As a finite element we use a triangular element with quadratic approximation of the displacement vector components and linear approximation of the rotation vector components.

3 ANALYTICAL SOLUTIONS OF TWO-DIMENSIONAL PROBLEMS OF ASYMMETRIC ELASTICITY THEORY

The analytical solutions are constructed in the context of the following example problems (Fig. 1):

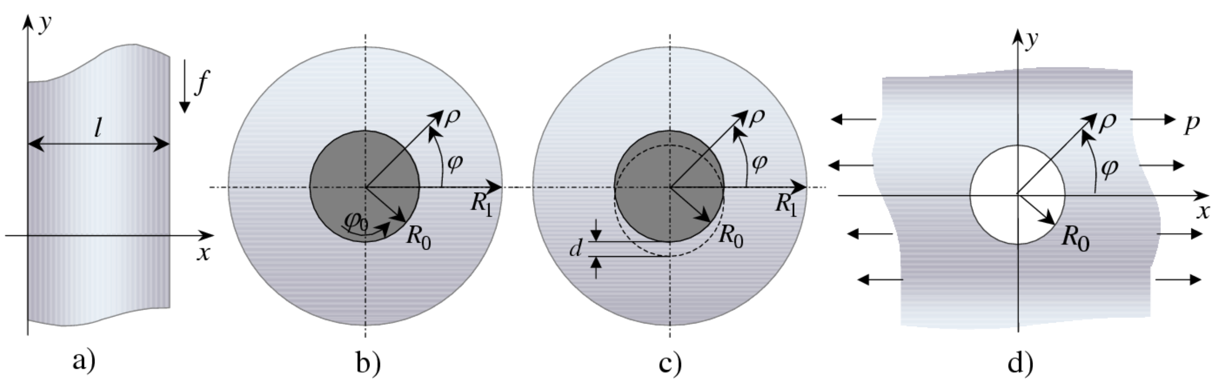

Figure 1: Computational schemes.

1. A plane infinite layer (plate) of width l subjected to mass forces of intensity f, acting along the axis $0y$ is in the state of equilibrium. The left ($x = 0$) and the right ($x = l$) edges are fixed:

$$\vec{u}|_{x=0} = \vec{\omega}|_{x=0} = \vec{0}, \quad \vec{u}|_{x=l} = \vec{\omega}|_{x=l} = \vec{0}. \qquad (9)$$

The solution in terms of the classical elasticity theory is given as [26]:

$$u_y^*(x) = \frac{fx}{2}(x - l). \qquad (10)$$

2. A plane ring rigidly fixed at the external contour $\rho = R_1$ is subjected to torsion due rotation of the internal contour $\rho = R_0$ by the angle φ_0:

$$u_\varphi\big|_{\rho=R_0} = \varphi_0 \cdot R_0, \quad \omega_z\big|_{\rho=R_0} = \varphi_0, \quad u_\varphi\big|_{\rho=R_1} = 0, \quad \omega_z\big|_{\rho=R_1} = 0. \tag{11}$$

The solution in terms of the classical elasticity theory is written as [26]:

$$u^*_\varphi(\rho) = \frac{R_0^2\varphi_0}{(1-R_0^2)\rho} - \frac{R_0^2\varphi_0}{1-R_0^2}\rho. \tag{12}$$

3. A plane ring rigidly fixed at the external contour $\rho = R_1$ is under shear deformation due to a rigid displacement of the internal contour $\rho = R_0$ by a magnitude d:

$$\begin{array}{ll} u_\rho\big|_{\rho=R_0} = -d\cos(\varphi), & u_\varphi\big|_{\rho=R_0} = d\sin(\varphi), \\ \omega_z\big|_{\rho=R_0} = 0, & u_\rho = u_\varphi = \omega_z\big|_{\rho=R_1} = 0. \end{array} \tag{13}$$

The solution of classical elasticity theory is also known [26]:

$$\begin{aligned} u^*_\rho(\rho,\varphi) = &\left(C_1^* + \frac{C_2^*}{\rho^2} + C_3^*\rho^2 + C_4^*\ln(\rho)\right)\cos(\varphi), \\ u^*_\varphi(\rho,\varphi) = &\left(-C_1^* + \frac{C_2^*}{\rho^2} + C_3^*\frac{3\lambda+5\mu}{\mu-\lambda}\rho^2 - C_4^*\left\{\ln(\rho) + \frac{\mu+\lambda}{3\mu+\lambda}\right\}\right)\sin(\varphi). \end{aligned} \tag{14}$$

The constants $C_1^*, \ldots, C_4^*$ are defined from the corresponding part of the boundary conditions (13).

4. An infinite plane weakened by a circular hole is under tensile stresses. The edge of the circular hole is free of the external stresses, and the plate at infinity is subjected to a tensile stress of constant intensity p in the direction of the axis Ox:

$$\begin{array}{lll} \sigma_{\rho\rho}\big|_{\rho=R_0} = 0, & \sigma_{\rho\varphi}\big|_{\rho=R_0} = 0, & \mu_{\rho z}\big|_{\rho=R_0} = 0, \\ \sigma_{\rho\rho}\big|_{\rho\to\infty} = p_\rho, & \sigma_{\rho\varphi}\big|_{\rho\to\infty} = p_\varphi, & \mu_{\rho z}\big|_{\rho\to\infty} = 0. \end{array} \tag{15}$$

In the framework of the classical elasticity theory this problem was first solved by Kirsch (1898), and later within slightly different approach by Muskhelishvili [27]:

$$\begin{aligned} u^*_\rho(\rho,\varphi) = &\left(C_1^*\rho + \frac{C_2^*}{\rho}\right) + \left(\frac{C_3^*}{\rho^3} + \frac{C_4^*}{\rho} + C_5^*\rho\right)\cos(2\varphi), \\ u^*_\varphi(\rho,\varphi) = &\left(\frac{C_3^*}{\rho^3} - C_4^*\frac{\kappa-1}{(\kappa+1)\rho} - C_5^*\rho\right)\sin(2\varphi). \end{aligned} \tag{16}$$

where:

$$C_1^* = \frac{p(\kappa-1)}{8}, \quad C_2^* = \frac{pR_0^2}{4}, \quad C_3^* = -\frac{pR_0^4}{4},$$
$$C_4^* = \frac{pR_0^2(\kappa+1)}{4}, \quad C_5^* = \frac{p}{4}, \quad C_6^* = 0.$$

The dimensionless quantity $\kappa = \dfrac{3\mu+\lambda}{\mu+\lambda}$.

The solutions to the boundary value problems a)-d) (Fig. 1) were obtained by applying the direct integration technique to the corresponding equilibrium equations.

Problems a)-d) (Fig. 1) are combined on the basis that they are most conveniently represented in the cylindrical coordinates (ρ, φ, z) in the form of truncated Fourier series:

$$\begin{aligned} \vec{u}(\rho,\varphi) = \quad & \{u_\rho(\rho,\varphi), u_\varphi(\rho,\varphi), 0\}, \\ & u_\rho(\rho,\varphi) = U^{(0)}(\rho) + \sum_{n=1}^{N} U^{(n)}(\rho)\cos(n\varphi), \\ & u_\varphi(\rho,\varphi) = V^{(0)}(\rho) + \sum_{n=1}^{N} V^{(n)}(\rho)\sin(n\varphi), \\ \vec{\omega}(\rho,\varphi) = \quad & \{0, 0, \omega_z(\rho,\varphi)\}, \\ & \omega_z(\rho,\varphi) = \omega^{(0)}(\rho) + \sum_{n=1}^{N} \omega^{(n)}(\rho)\sin(n\varphi). \end{aligned} \tag{17}$$

Substitution of relation (17) into equation (5) gives a sequence of systems of ordinary differential equations for the functions $U^{(n)}(\rho)$, $V^{(n)}(\rho)$ and $\omega^{(n)}(\rho), n = 0, 1, ..., n$. Passing over the details of constructing general solutions for these functions [28], we write them:

for zero harmonic $n = 0$ as

$$\begin{gathered} U^{(0)}(\rho) = C_1^{(0)}\rho + C_2^{(0)}\frac{1}{\rho}, \\ V^{(0)}(\rho) = C_3^{(0)}\rho + \frac{C_4^{(0)}}{\rho} - C_5^{(0)}\frac{I_1(2A\rho)}{2A^2} - C_6^{(0)}\frac{K_1(2A\rho)}{2A^2}, \\ \omega^{(0)}(\rho) = C_3^{(0)} - C_5^{(0)}\frac{BI_0(2A\rho)}{2A} + C_6^{(0)}\frac{BK_0(2A\rho)}{2A}. \end{gathered} \tag{18}$$

for the first harmonic $n = 1$ as

$$U^{(1)}(\rho) = C_1^{(1)} + \frac{C_2^{(1)}}{\rho^2} + C_3^{(1)}\rho^2 + C_4^{(1)}\ln(\rho) + C_5^{(1)}\frac{I_1(2A\rho)}{\rho} + C_6\frac{I_1(2A\rho)}{\rho},$$

$$\begin{gathered} V^{(1)}(\rho) = -C_1^{(1)} + \frac{C_2^{(1)}}{\rho^2} + C_3^{(1)}\frac{3\lambda+5\mu}{\mu-\lambda}\rho^2 - C_4^{(1)}\left\{\ln(\rho) + \frac{\mu+\lambda}{3\mu+\lambda}\right\} + \\ +C_5^{(1)}\left\{\frac{I_1(2A\rho)}{\rho} - 2AI_0(2A\rho)\right\} + C_6^{(1)}\left\{\frac{K_1(2A\rho)}{\rho} + 2AK_0(2A\rho)\right\}, \end{gathered} \tag{19}$$

$$w^{(1)}(\rho) = C_3^{(1)}\frac{8\mu+4\lambda}{\mu-\lambda}\rho - C_4^{(1)}\frac{2\mu+\lambda}{(3\mu+\lambda)\rho} - 2A^2BI_1(2A\rho)C_5^{(1)} - 2A^2BK_1(2A\rho)C_6^{(1)}.$$

and for higher harmonics $n \geq 2$ as

$$U^{(n)}(\rho) = \frac{C_1^{(n)}}{\rho^{(n+1)}} + \frac{C_2^{(n)}}{\rho^{(n-1)}} + C_3^{(n)}\rho^{(n-1)} + C_4^{(n)}\rho^{(n+1)} + \frac{C_5^{(n)}}{\rho} I_n(2A\rho) + \frac{C_6^{(n)}}{\rho} K_n(2A\rho),$$

$$\begin{gathered} V^{(n)}(\rho) = \frac{C_1^{(n)}}{\rho^{(n+1)}} + C_2^{(n)} \frac{n\lambda - 2\lambda + n\mu - 4\mu}{n\lambda + n\mu + 2\mu} \frac{1}{\rho^{(n-1)}} - C_3^{(n)}\rho^{(n-1)} - \\ -C_4^{(n)} \frac{n\lambda + 2\lambda + n\mu + 4\mu}{n\lambda + n\mu - 2\mu} \rho^{(n+1)} + C_5^{(n)} \left\{ \frac{I_n(2A\rho)}{\rho} - \frac{2A}{n} I_{n-1}(2A\rho) \right\} + \\ +C_6^{(n)} \left\{ \frac{K_n(2A\rho)}{\rho} + \frac{2A}{n} K_{n-1}(2A\rho) \right\}, \end{gathered} \tag{20}$$

$$\begin{gathered} \omega^{(n)}(\rho) = C_2^{(n)} \frac{2n\lambda - 2\lambda + 4n\mu - 4\mu}{n\lambda + n\mu + 2\mu} \frac{1}{\rho^n} - C_4^{(n)} \frac{2n\lambda + 2\lambda + 4n\mu + 4\mu}{n\lambda + n\mu - 2\mu} \rho^n - \\ -C_5^{(n)} \frac{2A^2B}{n} I_n(2A\rho) - C_6^{(n)} \frac{2A^2B}{n} K_n(2A\rho). \end{gathered}$$

In relations (18)-(20) $I_n(\rho)$ is the modified Bessel function of the first kind in the limit tending to infinity at $\rho \to \infty$, and $K_n(\rho)$ is the modified Bessel function of the second kind or the MacDonald function in the limit tending to zero at $\rho \to \infty$, $C_1^{(n)}, ...C_6^{(n)}$ are the constants obtained from the boundary conditions.

3.1 Problem of layer (plate) shear deformation

The problem of layer (plate) shear deformation has the following solution:

$$\begin{gathered} u_y(x) = C_1 + C_2 x + C_3 e^{2Ax} + C_4 e^{-2Ax} + \frac{f}{2} x^2, \\ \omega_z(x) = \frac{C_2}{2} + C_3 ABe^{2Ax} - C_4 ABe^{-2Ax} + \frac{f}{2} x, \\ \gamma_{xy}(x) = \frac{C_2}{2} - C_3 A(B-2)e^{2Ax} + C_4 A(B-2)e^{-2Ax} + \frac{f}{2} x, \\ \gamma_{yx}(x) = \frac{C_2}{2} + C_3 ABe^{2Ax} - C_4 ABe^{-2Ax} + \frac{f}{2} x, \\ \chi_{xz}(x) = 2C_3 A^2 Be^{2Ax} + 2C_4 A^2 Be^{-2Ax} + \frac{f}{2}, \\ \sigma_{xy}(x) = fx + C_2, \quad \sigma_{yx}(x) = fx + C_2 + 4C_3 Ae^{2Ax} - 4C_4 Ae^{-2Ax}, \\ \mu_{xz}(x) = 2C_3 e^{2Ax} + 2C_4 e^{-2Ax} + \frac{f}{2A^2B}, \quad \mu_{zx}(x) = C\mu_{xz}(x). \end{gathered} \tag{21}$$

Boundary conditions (9) give a system of linear algebraic equations, from which the constants $C_1, \ldots, C_4$ are defined.

Here and in the following, A, B, C are dimensionless complexes (7), and all quantities are non-dimensional. The linear quantities are referred to characteristic dimensions of the problem l, the stresses σ_{ij} to the shear modulus μ, and the couple-stresses μ_{ij} are multiplied by $\frac{l}{\mu}$.

The numerical simulation shows that in the experiments intended the maximum axial displacement $u_y(l/2)$ can be used as a measurable parameter of the problem. The degree of difference between the solution of the couple-stress theory (21) and that of the classical theory (10) is defined by

$$\delta_1 = \left|\frac{u_y(l/2) - u_y^*(l/2)}{u_y^*(l/2)}\right| \cdot 100\%. \tag{22}$$

Herinafter $(.)^*$ denotes quantities, referring to the clasical elasticity theory.

3.2 Problem of ring torsion

The solution of the ring torsion problem is obtained under the constraint of retaining one term in the series (17) at $n = 0$ and taking into account that $u_\rho(\rho) = 0$:

$$\begin{gathered}
u_\varphi(\rho) = C_1\rho + C_2\frac{1}{\rho} - C_3\frac{I_1(2A\rho)}{2A^2} - C_4\frac{K_1(2A\rho)}{2A^2}, \\
\omega_z(\rho) = C_1 - C_3\frac{BI_0(2A\rho)}{2A} + C_4\frac{BK_0(2A\rho)}{2A}, \\
\sigma_{\rho\varphi}(\rho) = -\frac{2C_2}{\rho^2} + C_3\frac{I_1(2A\rho)}{2A^2\rho} + C_4\frac{K_1(2A\rho)}{2A^2\rho}, \\
\sigma_{\varphi\rho}(\rho) = -\frac{2C_2}{\rho^2} + C_3\left(\frac{I_1(2A\rho)}{2A^2\rho} - \frac{2I_0(2A\rho)}{A}\right) + C_4\left(\frac{K_1(2A\rho)}{2A^2\rho} + \frac{2K_0(2A\rho)}{A}\right), \\
\mu_{\rho z}(\rho) = -C_3\frac{I_1(2A\rho)}{2A^2} - C_4\frac{K_1(2A\rho)}{2A^2},
\end{gathered} \tag{23}$$

Here the value of the external contour radius R_1 is taken as a characteristic linear dimension.

The constants $C_1, \ldots, C_4$ are defined from the boundary conditions (11) for particular values of dimensionless quantities A and B.

Based on the results of numerical simulation we found that in this problem the torque at the internal contour M can be used as experimentally measurable parameter:

$$M = \int_0^{2\pi} \sigma_{\rho\varphi}(R_0)R_0^2 d\varphi + \int_0^{2\pi} \mu_{\rho z}(R_0)R_0 d\varphi, \tag{24}$$

and the degree of difference between the couple-stress (23) and classical (12) solutions is defined by

$$\delta_2 = \left|\frac{M - M^*}{M^*}\right| \cdot 100\%. \tag{25}$$

3.3 Problem of ring deformation

The solution of the ring deformation problem is obtained under the requirement of retaining one term in the series (17) for $n = 1$ (26):

$$
\begin{gathered}
u_\rho(\rho,\varphi) = \cos(\varphi)\left(C_1 + \frac{C_2}{\rho^2} + C_3\rho^2 + C_4\ln(\rho) + C_5\frac{I_1(2A\rho)}{\rho} + C_6\frac{K_1(2A\rho)}{\rho}\right),\\
u_\varphi(\rho,\varphi) = \sin(\varphi)\left(-C_1 + \frac{C_2}{\rho^2} + C_3\frac{3\lambda+5\mu}{\mu-\lambda}\rho^2 - C_4\left\{\ln(\rho) + \frac{\mu+\lambda}{3\mu+\lambda}\right\} + \right.\\
\left. + C_5\left\{\frac{I_1(2A\rho)}{\rho} - 2AI_0(2A\rho)\right\} + C_6\left\{\frac{K_1(2A\rho)}{\rho} + 2AK_0(2A\rho)\right\}\right),\\
\omega_z(\rho,\varphi) = \sin(\varphi)\left(C_3\frac{8\mu+4\lambda}{\mu-\lambda}\rho - \frac{C_4(2\mu+\lambda)}{(3\mu+\lambda)\rho} - 2A^2BI_1(2A\rho)C_5 - 2A^2BK_1(2A\rho)C_6\right),\\
\sigma_{\rho\rho}(\rho,\varphi) = \cos(\varphi)\left(-C_2\frac{4}{\rho^3} + C_3\frac{4(\mu+\lambda)}{\mu-\lambda}\rho + C_4\frac{2(3\mu+2\lambda)}{(3\mu+\lambda)\rho} + C_5S^{(5)}_{\rho\rho}(\rho) + C_6S^{(6)}_{\rho\rho}(\rho)\right),\\
\sigma_{\rho\varphi}(\rho,\varphi) = \sin(\varphi)\left(-C_2\frac{4}{\rho^3} + C_3\frac{4(\mu+\lambda)}{\mu-\lambda}\rho - C_4\frac{2\mu}{(3\mu+\lambda)\rho} + C_5S^{(5)}_{\rho\varphi}(\rho) + C_6S^{(6)}_{\rho\varphi}(\rho)\right),
\end{gathered}
\tag{26}
$$

where

$$
\begin{aligned}
S^{(5)}_{\rho\rho}(\rho) &= \frac{4A}{\rho}I_0(2A\rho) - \frac{4}{\rho^2}I_1(2A\rho), \quad S^{(6)}_{\rho\rho}(\rho) = -\frac{4A}{\rho}K_0(2A\rho) - \frac{4}{\rho^2}K_1(2A\rho),\\
S^{(5)}_{\rho\varphi}(\rho) &= \frac{4A}{\rho}I_0(2A\rho) - \frac{4}{\rho^2}I_1(2A\rho), \quad S^{(6)}_{\rho\varphi}(\rho) = -\frac{4A}{\rho}K_0(2A\rho) - \frac{4}{\rho^2}K_1(2A\rho).
\end{aligned}
$$

The radius of the external contour R_1 is taken as a characteristic linear dimension.

The constants $C_1, \ldots, C_6$ are evaluated from the boundary conditions (13).

The results of numerical simulation have demonstrated that the degree of response of the internal contour F_y can be conveniently used as a measurable quantity:

$$
F_y = \int\limits_0^{2\pi} \left(\sigma_{\rho\varphi}(R_0,\varphi)\sin(\varphi) + \sigma_{\rho\rho}(R_0,\varphi)\cos(\varphi)\right) R_0 d\varphi, \tag{27}
$$

and the degree of difference between the couple-stress (26) and classical (14) solutions is defined by

$$
\delta_3 = \left|\frac{F_y - F_y^*}{F_y^*}\right| \cdot 100\%; \tag{28}
$$

3.4 Problem on uniaxial tension of an infinite plate weakened in the center by a circular hole

The solution to the problem is obtained under the requirement of retaining two terms in the series (17) at $n = 0$ (23) and $n = 2$ (29):

$$
\begin{aligned}
u_\rho(\rho,\varphi) &= C_1\rho + \frac{C_2}{\rho} + \left(\frac{C_3}{\rho^3} + \frac{C_4}{\rho} + C_5\rho + C_6\rho^3 + C_7U_7(\rho) + C_8U_8(\rho)\right)\cos(2\varphi),\\
u_\varphi(\rho,\varphi) &= \left(\frac{C_3}{\rho^3} - C_4\frac{\kappa-1}{(\kappa+1)\rho} - C_5\rho - C_6\frac{\kappa+3}{\kappa-3}\rho^3 + C_7V_7(\rho) + C_8V_8(\rho)\right)\sin(2\varphi),\\
\omega_z(\rho,\varphi) &= \left(\frac{C_4}{\rho^2} - C_6\frac{3(\kappa+1)}{3-\kappa}\rho^2 + C_7\omega_7(\rho) + C_8\omega_8(\rho)\right)\sin(2\varphi),
\end{aligned} \tag{29}
$$

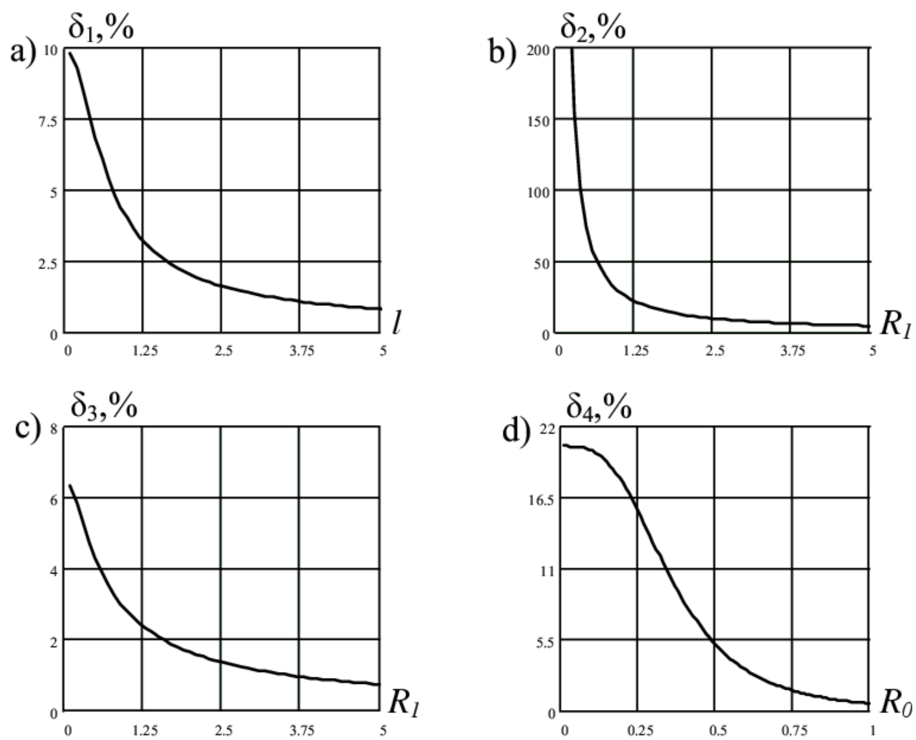

Figure 2: Difference between couple-stress and classical solutions.

The radius of the circular hole R_0 is selected as a linear characteristic dimension.

The constants are defines from the boundary conditions (15).

The numerical simulation has shown that parameter D, characterizing the degree of distortion of the circular hole boundary is convenient to use as an experimentally measurable parameter:

$$
D = \left|\frac{u_\rho(R_0,0)}{u_\rho(R_0,\frac{\pi}{2})}\right|, \tag{30}
$$

and the degree of difference between the couple-stress (29) and classical (16) solutions is defined by

$$\delta_4 = \left|\frac{D - D^*}{D^*}\right| \cdot 100\%. \qquad (31)$$

In Fig. 2, the introduced relative parameters δ_1 (31), δ_2 (25), δ_3 (28), δ_4 (31) are given versus the corresponding characteristic linear dimension of the problem. In the numerical analysis, the physical parameters were assumed to have the following values [21]: $\lambda = 2.096 \cdot 10^9 \frac{N}{m^2}$, $\mu = 1.033 \cdot 10^9 N/m^2$, $\alpha = 1.148 \cdot 10^8 N/m^2$, $\gamma = 4.1 \cdot 10^6 N$, $\varepsilon = 1.312 \cdot 10^5 N$. The plots b) and c) were constructed for a fixed value of $R_0 = 0.1$, characterizing the dimensionless radius of the internal contour and referred to characteristic dimension R_1, and the characteristic dimension in this case was chosen such that $R_1 > R_0$.

A comparison of the obtained four plots (Fig. 2) shows that the most informative are the problem of the plane ring torsion (b) and the problem on uniaxial tension of an infinite plate with a circular hole (d). As one might expect the informative value of shear deformation problem (a) proves to be the least due to the absence of large stress gradients. This comparison provides guidelines for realization of experiments capable of revealing couple-stress response of the examined media.

4 NUMERICAL SOLUTIONS OF TWO-DIMENSIONAL PROBLEMS OF ASYMMETRIC ELASTICITY THEORY

4.1 Problem of extension of a plate with five holes

Consider extension of a square plate with five circular holes located as shown in Fig. 3.a. The radii of all holes are equal. The external dimension of the plate is $L = 20 \cdot (3R + d)$, where d is the distance between the holes.

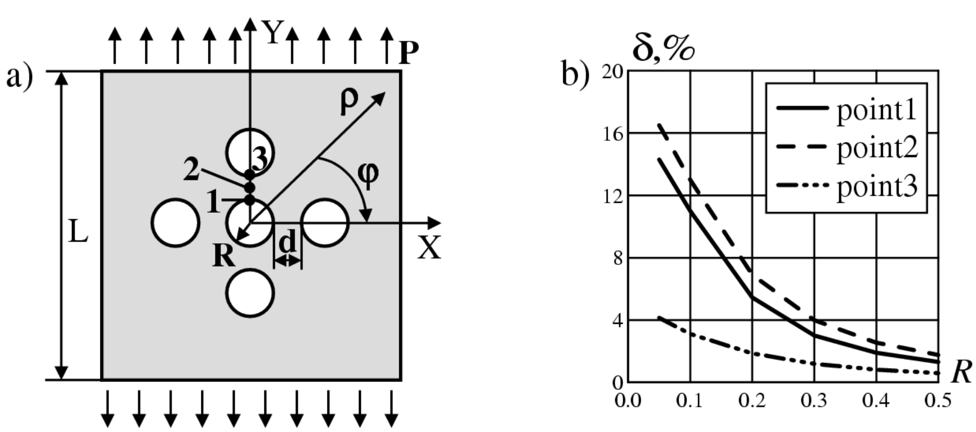

Figure 3: Extension of a plate with five holes (a); difference between couple-stress and classical solutions δ for different values of the hole radii (b), $d = R$.

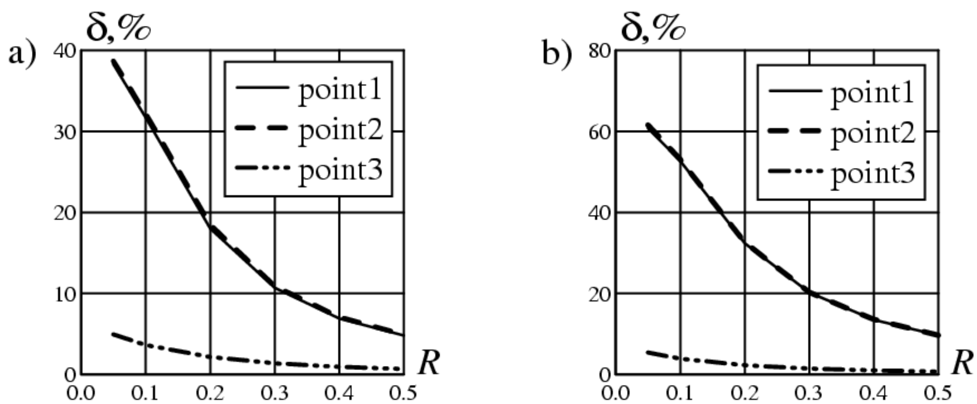

Figure 4: Difference between couple-stress and classical solutions δ for different values of the hole radii; $d = 0.5R$ (a), $d = 0.25R$ (b).

By virtue of symmetry a numerical analysis of the stress-strain state was made for 1/4 part of the plate. The faces lying on the X- and Y- axes are assumed to satisfy the symmetry conditions:

$$u_n|_{x=0} = u_n|_{y=0} = \omega_z|_{x=0} = \omega_z|_{y=0} = 0. \tag{32}$$

where u_n is the normal component of the displacement vector $\vec{u}$.

An acceptable variant of discretization was determined in the process of numerical experiment dealing with a comparison of computation results obtained on different finite-element grids. Discretization of the computational domain is schematically represented in Fig. 5.a.

All computations were made for the following values of the material physical constants [21]: $\lambda = 2.096 \cdot 10^9 \dfrac{N}{m^2}$, $\mu = 1.033 \cdot 10^9 \dfrac{N}{m^2}$, $\alpha = 1.148 \cdot 10^8 \dfrac{N}{m^2}$, $\gamma = 4.1 \cdot 10^6 N$, $\varepsilon = 1.312 \cdot 10^5 N$. The value of loading is $P = 1N$.

The difference between numerical couple-stress and classical solutions is defined by:

$$\delta = \left| \frac{u_y^m - u_y^{cl}}{u_y^{cl}} \right| \cdot 100\%. \tag{33}$$

where u_y^m is the solution of asymmetric elasticity theory (couple-stress solution), u_y^{cl} is the solution of symmetric elasticity theory (classical solution).

Computations were performed for different values of radius R and different values of spacing between the holes d. Some results of computation are given in Fig. 3b, Fig. 4.a and Fig. 4.b.

From the analysis of solutions obtained for plates with one and five holes we can draw the following conclusions:

— with increase of the number of holes the difference between the couple-stress and classical solutions becomes more drastic;

— the difference between the classical and couple-stress solutions increases as the inter-hole spacing decreases.

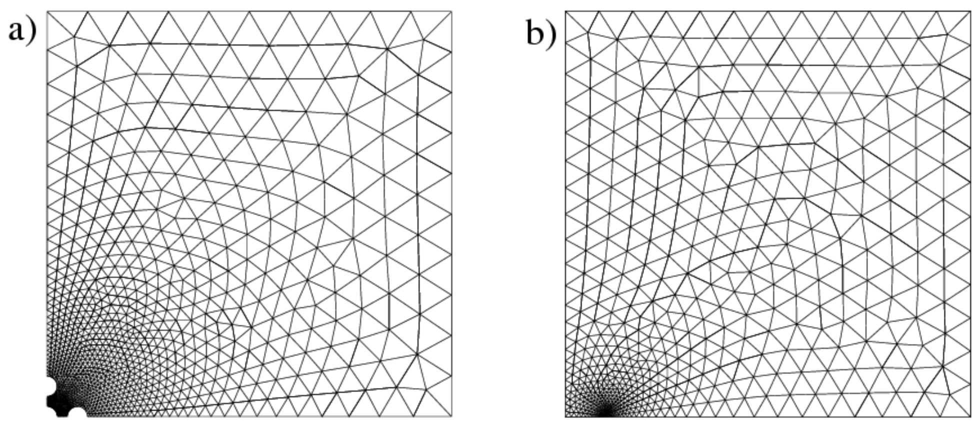

Figure 5: Discretization of computational domain for extension of a plate with five holes (a), $N = 2278$; for extension of a plate with a crack (b), $N = 1121$.

4.2 Problem of extension of a plate with a crack

Consider extension of a square plate with a crack located in the center (Fig. 6.a). The faces lying on the X- and Y- axes are assumed to satisfy the symmetry conditions (32).

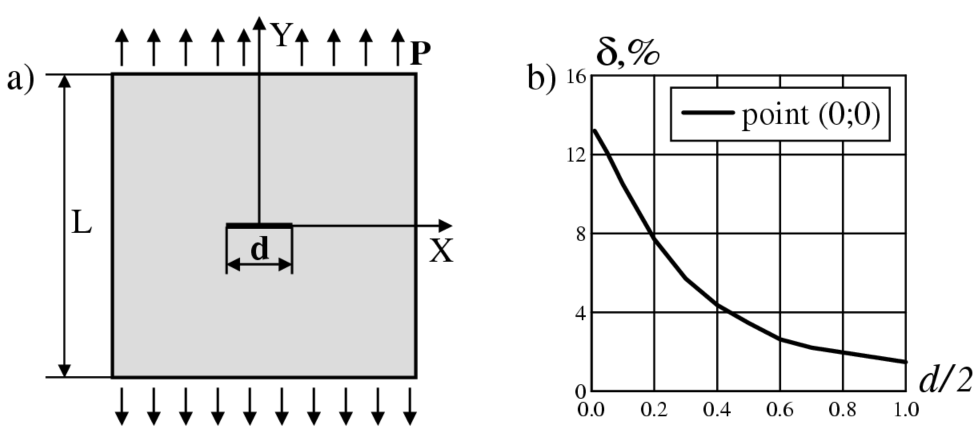

Figure 6: Extension of a plate with a crack (a); difference between couple-stress and classical solutions δ for various crack sizes d at the point (0;0) (b).

Numerical calculation was made on the finite-element grid with increased mesh density in the vicinity of the crack edge (Fig. 5.b).

Physical constants of the material are given the same values as in the previous problem.

In these calculations, the difference between the classical and couple-stress solutions was defined in terms of the displacement u_y at point $(X = 0; Y = 0)$.

The results of comparison given in Fig. 6.b, indicate that the crack opening obtained from the asymmetric elasticity solution is greater than that of the classical elastic solution.

5 METHOD OF SENSITIVITY ANALYSIS FOR PROBLEMS ON IDENTIFICATION OF MATERIAL PHYSICAL CONSTANTS IN ASYMMETRIC ELASTICITY THEORY

It is suggested to use the techniques of the sensitivity analysis to estimate the informative value of the examined asymmetric elasticity problems for experiments establishing mechanical characteristics of the material under consideration.

Let $z \in R^n$ be the vector of full (global) displacements and rotations the vector of the structure state; $b = (b_1, \cdots, b_k)^T$ be the vector of design variables, i.e $z = z(b)$. Physical constants of the couple-stress medium (4) are used as design variables.

The objective of the analysis is to define the dependence of the function $\Psi = \Psi(b, z(b))$ on the design variables, that is, $\frac{d\Psi}{db}$ or derivatives of sensitivity.

Consider the sensitivity analysis with reference to static problems of the asymmetric elasticity theory.

The behavior of structure under static loads can be described by the following equation:

$$K(b)z = F(b). \tag{34}$$

where $K(b)$ is the reduced global stiffness matrix, $F(b)$ is the reduced load. In the present case, the notion of "reduced" quantity implies that boundary conditions are taken into account.

Differentiability of the solution z with respect to design variables is proved by the theorem of implicit function, which on the assumption that all quantities entering $K(b)$ and $F(b)$, are s times differentiable with respect to design variables ensures that the solution of the equation (34) $z = z(b)$ is also s times continuously differentiable.

Using the rule of complex function differentiation and notations of matrix calculus we can calculate a full derivative of Ψ with respect to b:

$$\frac{d\Psi}{db} = \frac{\partial\Psi}{\partial b} + \frac{\partial\Psi}{\partial z}\frac{\partial z}{\partial b}. \tag{35}$$

Let us differentiate both parts of the equation (34) wit respect to b:

$$K(b)\frac{\partial z}{\partial b} = -\frac{\partial}{\partial b}\left(K(b)\tilde{z}\right) + \frac{\partial F(b)}{\partial b}. \tag{36}$$

here and in the following, the symbol $\sim$ denotes variable which in the process of partial differentiation must be constant.

Since $K(b)$ is non-degenerated, from the equation (36) we have:

$$\frac{\partial z}{\partial b} = K^{-1}(b)\left[\frac{\partial F(b)}{\partial b} - \frac{\partial}{\partial b}\left(K(b)\tilde{z}\right)\right]. \tag{37}$$

Substitution of (37) into (35), gives:

$$\frac{d\Psi}{db} = \frac{\partial\Psi}{\partial b} + \frac{\partial\Psi}{\partial z}K^{-1}(b)\left[\frac{\partial}{\partial b}\{F(b) - K(b)\tilde{z}\}\right]. \tag{38}$$

One of the drawbacks of the proposed approach is the need of calculating $K^{-1}(b)$. We can use two ways to overcome this problem. First, we can solve numerically the equation (37) for $\frac{dz}{db}$ and substitute the obtained result in (35) – the direct differentiation method.

Second, we can obtain sensitivity derivatives – the method of conjugate variables used in this work. It involves determination of the conjugate variable ξ:

$$\xi \equiv \left[\frac{\partial \Psi}{\partial z} K^{-1}(b)\right]^T = K^{-1}(b) \frac{\partial \Psi^T}{\partial z}. \tag{39}$$

Instead of determining ξ from equation (39), including $K^{-1}(b)$, we multiply both parts of this equation by $K(b)$ and obtain a conjugate equation:

$$K(b)\xi = \frac{\partial \Psi^T}{\partial z}. \tag{40}$$

Then, substituting ξ from (40) into (38) with account of (39) we define:

$$\frac{d\Psi}{db} = \frac{\partial \Psi}{\partial b} + \frac{\partial}{\partial b}\left[\tilde{\xi}^T F(b) - \tilde{\xi}^T \left(K(b)\tilde{z}\right)\right]. \tag{41}$$

Thus, in the sensitivity analysis we first find the solution of the static problem, which is then analyzed to establish the response of the objective function Ψ to variation of the vector b by δb.

The components of the vector $\frac{d\Psi}{db}$ are the derivatives of the function sensitivity to corresponding design variables.

Let $\vec{l}$ denote the sensitivity vector, the component of which l_i is a derivative with respect to the i-th variable. In particular, if $l_i > 0$ ($l_i < 0$), then an increase (decrease) of the design variable leads at minimization to a decrease (increase) of the function Ψ. The order of magnitude of different sensitivity coefficients l_i allows us to determine which of the design variables have essential effect on Ψ, and which of them are insignificant.

For a displacement vector we choose the vector $b = (\lambda, \mu, \alpha, \gamma, \varepsilon)$; for an objective function we take the function:

$$\Psi = \sum_{i=1}^{N}\left[\alpha_1\left(u^c_{xi} - u^e_{xi}\right)^2 + \alpha_2\left(u^c_{yi} - u^e_{yi}\right)^2 + \alpha_3\left(\omega^c_i - \omega^e_i\right)^2\right]. \tag{42}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the weight coefficients; symbols c and e denote the values of displacements and rotations in calculation of the model constants of the material $\lambda^c, \mu^c, \alpha^c, \gamma^c, \varepsilon^c$, and "experimental" model constants $\lambda^e, \mu^e, \alpha^e, \gamma^e, \varepsilon^e$; N is the number of points, at which the values of the objective function are calculated (42).

In studying sensitivity of the structure to variation of any of its parameters it is most convenient to operate with normalized vectors of sensitivity. Normalization is performed in the following way. Let $\vec{l} = (l_1, l_2, l_3, l_4, l_5)$ be the sensitivity vector, in which $l_1 = \frac{d\Psi}{d\lambda}$,

$l_2 = \frac{d\Psi}{d\mu}$, $l_3 = \frac{d\Psi}{d\alpha}$, $l_4 = \frac{d\Psi}{d\gamma}$, $l_5 = \frac{d\Psi}{d\varepsilon}$. Then the components of the normalized vector are defined by the formula:

$$l_i^n = \frac{l_i}{\sqrt{\sum\limits_{i=1}^{5} l_i^2}}. \tag{43}$$

With the aim to find the most informative experimental scheme (from the viewpoint of the material couple-stress response) we have tested the operation of the constructed finite element algorithm for evaluation of coefficients of the sensitivity analysis by solving a number of example problems:

1. Problem of extension of a plate with one hole (Fig. 1.d);
2. Problem of extension of a plate with five holes (Fig. 3.a);
3. problem of extension of a plate with a crack (Fig. 6.a).

The values of "experimental" physical constants were taken from the work [21]: $\lambda = 2.096 \cdot 10^9 \dfrac{N}{m^2}$, $\mu = 1.033 \cdot 10^9 \dfrac{N}{m^2}$, $\alpha = 1.148 \cdot 10^8 \dfrac{N}{m^2}$, $\gamma = 4.1 \cdot 10^6 N$, $\varepsilon = 1.312 \cdot 10^5 N$.

A comparison of the obtained sensitivity coefficients allows us to make the following conclusions. For example, in the problem on extension of a plate with five holes, out of the components of the sensitivity vector l_3, l_4, l_5, the component l_3 is most significant. Thus, the experiment based on this problem is most appropriate for identification of the parameter α. With reference to parameters γ and ε the problems considered give rather close values of the corresponding components of the sensitivity vector.

6 CONCLUSIONS

- The analytical and numerical solutions to a number of two-dimensional boundary value problems of the asymmetric elasticity theory have been considered.
- The most demonstrative examples of difference between the solutions of symmetric and asymmetric elasticity theory have been found.
- The parameters of material that can be measured in experiments designed for detecting the effects of the couple-stress behavior of a medium have been identified.
- The obtained values of the sensitivity coefficients have been used to estimate the informative value of the examined problems from the viewpoint of realization of experiments allowing one to identify mechanical constants of the asymmetric elasticity theory.

This work was supported by Russian Foundation of Fundamental Research under project 03–01–00561 and in part by Award No. PE–009–0 of the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).

REFERENCES

[1] H. Schaefer. Das Cosserat-Kontinuum. ZAMM, V.47, No.8, P.485-498 (1967).

[2] R. D. Mindlin. Stress function for a cosserat continuum. Int. J. Engng. Sci, V.1, P.265-271, (1965).

[3] W. Nowacki. Theory of elasticity (Russian translation). Mir, Moscow, (1975).

[4] V. A. Palmov. Basic equations of asymmetric theory of elasticity (in Russian). Prikl. Mat. Mekh., V.28, No.3, P.401-408 (1964).

[5] V. A. Palmov. Plane problems of asymmetric elasticity theory (in Russian). Prikl. Mat. Mekh., V.28, No.6, P.1117-1120 (1964).

[6] R. A. Toupin. Elastic materials with couple-stress. Arch. Ration. Mech. and Analysis, V.11, No.5, P.385-413 (1962).

[7] W. T. Koiter. Couple-stress in the theory of elasticity. Proc. Koenicl. Acad. Wet, V.B67, No.17 (1964).

[8] R. D. Mindlin. Influence of couple-stress on stress concentrations. Experimental Mechanics, V.3, No.1, P.1-7 (1963).

[9] N. F. Morozov. Selected two-dimensional problems of the elasticity theory. L.: Leningrad University Press (1978).

[10] G. N. Savin. Mechanics of deformable solids (in Russian). Nauk. Dumka,Kiev (1979).

[11] A. I. Kalandia. Mathematical methods of two-dimensionsional elasticity (in Russian). Nauka, Moscow (1973).

[12] A. C. Eringen. Theory of micropolar elasticity. In H.Liebowiz (ed.), Fracture, Academic Precc, V.2, P.621 (1968).

[13] A. Anthonie. Effect of couple-stress on the elastic bending of beams. Int. J. of Solid and Structures. V.37, P.1003-1018 (2000).

[14] D. B. Bogy and E. Sternberg. The effect of couple-stress on the corner singularity due to an asymmetric shear loading. Int. J. of Solid and Structures, V.4, P.159-174 (1968).

[15] Y. C. Hsu and W. J. Wang. Couple-stress effects near an interior hole of an infinite elastic plane subjected to a concentated force. J. Franklin Inst, V.295, No.5, P.411-421 (1973).

[16] Shoichi Kobayashi and Fukui Takio. Effects of couple stresses on stress distribution in a ring test specimen. Mem. Fac. Eng. Kyoto Univ, V.33, No.4, P.233-242 (1971).

[17] R. D. Gauthier and W.E.Jahsman. A quest for micropolar elastic constants. Trans. ASME, V.E42, No.2, P.369-374 (1975).

[18] V. Erofeev, V. Rediushkin. Observation of elastic wave dispersion in a grain composite and mathematical model for its description (in Russian). Acoustic J., V. 38, No.6, P.1116-1117(1992).

[19] R. D. Gauthier and W. E. Jahsman. A quest for micropolar elastic constants: Part 2, Arch. Mech, V.33, No.5, P.717-737 (1981).

[20] Dynamics and stability of laminar composite materials (in Russian). Nauk. Dumka, Kiev (1992).

[21] R. Lakes. Cosserat micromechanics of structured media experimental methods. Third technical conference "Proceedings of the American society for composites", September 25-29, Seatle, Washington. P.505-516 (1988).

[22] Ya. Ya. Rushitskiy. Elements of mixture theory (In Russian). Nauk.Dumka, Kiev (1991).

[23] M. Onami, S. Ivasimidzu, K. Genka, K. Siodzava, K. Tanaka. Introduction to micromechanics (Russian translation). Metallurgy, Moscow (1987).

[24] E. Kroener. On the physical reality of torque stresses in continuum mechanics. Int. J. Eng. Sci, V.1, P.261 (1963).

[25] V. Kupradze, T. Gegelia, M. Basheleishvili, T. Burchuladze. Three-dimensional problems in the mathematical theory of elasticity and thermo-elasticity (in Russian). Nauka, Moscow (1976).

[26] A. I. Lur'e. Theory of elasticity (in Russian). Hauka, Moscow (1970).

[27] N. I. Muskhelishvili. Some basic problems of the mathematical theory of elasticity (in Russian). Nauka, Moscow, (1966). English translation: Noordhoff, Groningen, Holland (1953).

[28] M. A. Kulesh, V. P. Matveenko, I. N. Shardakov. Parametric analysis of analytical solutions to one- and two-dimensional problems in couple-stress theory of elasticity. Z. Angew. Math. Mech, 83(4), P.238-248, (2003).