Rayleigh waves in the isotropic and linear, reduced Cosserat continuum

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Abstract

We continue in this paper the work on the modeling of rocks and soils in terms of the linear, elastic, reduced Cosserat continuum. The reduced Cosserat continuum is a continuum where each point possesses rotational degrees of freedom. Furthermore, the medium resists to rotation as well as to translation, while the couple stress is zero. The stress tensor is asymmetric. The objective of the model is to take into account the microstructure of rocks and soils which influences wave propagation. It was first suggested by Shwartz, Johnson, and Feng in 1984 to describe granular materials. Wave propagation in an unbounded 3D Cosserat continuum was investigated by Grekova and Herman in 2003–2005. In this work, we consider the Rayleigh wave for the isotropic case, using analytical and numerical methods. Instead of a straight line in the classical medium, we obtain two dispersion curves. The polarization differs both from the case of the classical medium and the case of a Cosserat continuum with couple stresses. For some frequency range, we observe a strong frequency dependence. There is a forbidden band of frequencies, lying below the analogous forbidden band for an unbounded medium. It indicates the possibility of localization phenomena. For the upper branch of the dispersion relation, there is also a forbidden domain of wave numbers: long waves may propagate only with one frequency. Far from the domain of frequencies where the microstructure influences the wave propagation, the medium behaves analogously to the classical one (as expected). We make a comparison with the classical medium and the Cosserat medium with couple stress for which the Rayleigh wave was investigated by Kulesh et al.

1 Introduction

This paper continues the series of previous works reported at APM conferences on the modelisation of rocks and soils in terms of the elastic reduced Cosserat continuum. The general idea of these works is to take into account the microstructure of
rocks and soils which influences the wave propagation, for some frequency range trapping the energy of the propagating wave by their proper rotational motion. Instead of attacking the detailed microstructural problem, we consider a phenomenological theory based on general fundamental laws. Firstly such a theory was suggested by Schwartz, Johnson, and Feng (1984) for modelling granular materials, and investigated in isotropic case by Grekova and Herman (2004) and generalized by Grekova and Herman (2005) for the case of weak local anisotropy. In this medium, each particle is an infinitesimal rigid body possessing rotational degrees of freedom, and the surrounding medium resists to its rotation making the stress tensor asymmetric. However, the couple stress is not present. Such a model is a kind of effective model accounting for frequency-dependent transmission and wave localisation, similar to those caused by wave scattering by heterogeneities. The particle of the medium may correspond to an inclusion, crack, or another heterogeneity with the surrounding medium attached to it, or in some cases to a real particle of a granular material with some average properties. For the wave propagation in an unbounded 3D medium we have observed in a certain frequency range strong dispersion and attenuation, and localisation for the shear wave in the isotropic case, and similar effects both for the compression and shear waves in case of the weak local anisotropy coupling shear-rotational and volumetric deformation. One of the most important characteristics of any medium is the propagation of the Rayleigh wave. To have a physical idea on the behaviour of the surface waves in our medium was the motivation for this work. We consider the simplest variant of isotropic case (with spherical inertia tensors). Even in this case the dispersive behaviour is very complex, and the analytical investigation is not easy, and still has to be completed. We observe (similar to the case of 3D unbounded medium, analysed in previous works) a forbidden band of frequencies, strong dispersive behaviour, and non-standard polarization.

Statement of the problem. Dispersion relation for the Rayleigh wave

We consider a linear elastic medium whose particles possess rotational degrees of freedom. The motion of each particle is described by the displacement vector \( u \) and rotation vector \( \theta \). We consider a medium where the elastic energy depends on the deformation tensor \( g = \nabla u + \theta \times E \), where \( E \) is the unit tensor. This means that a particle may feel its neighbourhood only as a continuum of mass points. For the isotropic case the constitutive equation of such a medium is

\[
\tau = \lambda \nabla \cdot u + 2\mu (\nabla u)^2 + 2\alpha (\nabla u + \theta \times E)^A,
\]

where \( \lambda \) and \( \mu \) are Lamé coefficients, \( \alpha \) is an elastic constant describing the resistance to rotation of the point-body with respect to the background. If \( \alpha = 0 \), the model reduces to classical elasticity. The balance of force and the balance of torque have the form:

\[
\nabla \cdot \tau + \rho K = \rho \ddot{u}, \quad \tau + \rho L = (I \cdot \dot{\theta})',
\]

where \( K \) and \( L \) are matrices related to the stress and couple stress, respectively.
Here, dot is the material time derivative, $\mathbf{\tau}_x = \tau_{mn} i_m \times i_n$, $\rho$ is the mass density of the medium, $\mathbf{I}$ the mass density of the inertia tensor, $\mathbf{K}$ and $\mathbf{L}$ are the densities of the external volume force and torque, respectively. We consider, for simplicity, the case of a spherical inertia tensor: $\mathbf{I} = I \mathbf{E}$.

Let us consider the case of zero external load: $\mathbf{K} = 0$, $\mathbf{L} = 0$. Using the constitutive equation for $\mathbf{\tau}$, we obtain the following form of the laws of dynamics

$$\begin{align*}
(\lambda + 2\mu) \nabla \nabla \cdot \mathbf{u} - (\mu + \alpha) \nabla \times (\nabla \times \mathbf{u}) + 2\alpha \nabla \times \mathbf{\theta} + \rho \mathbf{K} &= \rho \ddot{\mathbf{u}}, \\
2\alpha \nabla \times \mathbf{u} - 4\alpha \mathbf{\theta} + \rho \mathbf{L} &= \rho \ddot{\mathbf{\theta}}.
\end{align*}$$

(3)

Consider the wave propagation along the surface of the free elastic half-space. Let $z$ be the vertical co-ordinate with the corresponding unit vector $\mathbf{i}_3$, $x, y$ — coordinates in the plane (axes $\mathbf{i}_1, \mathbf{i}_2$). We look for the solution of (3) in the form of $\mathbf{u} = \mathbf{U}(z) e^{i(kx - \omega t)}, \mathbf{\theta} = \Theta(z) e^{i(kx - \omega t)}$. The equation of motion (3) then separates in two independent systems: the system for $\mathbf{U}_x(z), \mathbf{U}_z(z), \Theta_y(z)$ and the system for $\mathbf{U}_y(z), \Theta_x(z), \Theta_z(z)$. To investigate Rayleigh-type wave in the medium we look for the solutions of the first system, decreasing with depth $z$. We obtain

$$\begin{align*}
\mathbf{U}_x(z) &= D_1 i \mathbf{e}^{-\nu_1 z} + D_2 \mathbf{e}^{-\nu_2 z}, \\
\mathbf{U}_z(z) &= -D_1 \mathbf{e}^{-\nu_1 z} + D_2 \mathbf{e}^{-\nu_2 z}, \\
\Theta_y(z) &= -D_2 \frac{2\alpha \omega^2}{1 - \omega^2/\omega_0^2} \mathbf{e}^{-\nu_2 z} \equiv D_2 \frac{\omega^2}{2c_S^2(1 - \omega^2/\omega_1^2)} \mathbf{e}^{-\nu_2 z},
\end{align*}$$

(4)

where

$$(\nu_1, \nu_2) = \sqrt{k^2 - \omega^2/c_p^2}, \quad (\nu_1, \nu_2) = \sqrt{k^2 - \omega^2/\omega_0^2}, \quad \omega_0^2 = \frac{4\alpha}{1}, \quad \omega_1^2 = \frac{\omega_0^2}{1 + \alpha/\mu}, \quad c_S^2 = \frac{\mu}{\rho}.$$  (5)

Applying the boundary condition $\mathbf{i}_3 \cdot \mathbf{\tau} = 0$, we obtain the dispersion relation for the Rayleigh wave:

$$4k^2 \nu_1 \nu_2 = (2k^2 - \omega^2/c_p^2)^2.$$  (6)

Here $\nu_1, \nu_2 \in \mathbb{R}$. Curves $\nu_1 = 0, \nu_2 = 0$, bounding the “allowed zone” for the Rayleigh wave, are the dispersion relations for the compression and shear-rotation plane waves in 3D, respectively (see Grekova and Herman, 2004). The “allowed zone” is shown in Figure 1. Here, $c_p = \sqrt{(\lambda + 2\mu)/\rho}$ is the velocity of the compression plane wave in 3D, $c_{S\alpha} = \sqrt{[\mu + \alpha]/\rho}$ is the velocity of the 3D shear plane wave for $\omega \to \infty$.

**Numerical analysis**

The numerical analysis of the dispersion relation (6) for the constants $\lambda = 28 \cdot 10^9, \mu = 4 \cdot 10^9, \alpha = 2 \cdot 10^9, I = 10^4$ (in S.I. units) is represented in Figure 2. The red
Analytical results

We may write down another form of the Rayleigh equation, introducing a new variable $x$ such that $k = \omega \sqrt{x/2/C_s}$:

$$a(\omega)x^3 + b(\omega)x^2 + 4x - 1 = 0,$$

where

$$a(\omega) = 2 \left( 1 - \frac{C_s^2}{C_p^2} + \frac{\omega^2}{(\lambda + 2\mu + \alpha) |\omega^2 - \omega_p^2|} \right),$$

$$b(\omega) = 2 \left( 2 \frac{C_s^2}{C_p^2} - 3 - \frac{2C_s^2 \omega^2}{C_p^2 |1 + \mu/\alpha| (\omega^2 - \omega_p^2)} \right),$$

$$c(\omega) = \sqrt{27a(\omega)^2 - 2b(\omega)^3 + 36a(\omega)b(\omega) + d(\omega)},$$

$$d(\omega) = 3 \sqrt{3a(\omega) \sqrt{27a(\omega)^2 - 4b(\omega)^3 - 16b(\omega)^2 + 72a(\omega)b(\omega) + 256a(\omega)}}.$$
Having resolved this cubic equation, we obtain

\[ k(\omega) = \frac{\omega}{C_s} \sqrt{\frac{\sqrt{3} c(\omega)}{12 a(\omega)} - \frac{b(\omega)}{6 a(\omega)} + \frac{\sqrt{2} (b(\omega)^2 - 12 a(\omega))}{6 a(\omega) c(\omega)}}. \] (7)

The dispersion relation (7) is complicated and depends on various parameters. We investigate analytically some its properties. One may show that provided \( \nu_1, \nu_2 \in \mathbb{R} \), the equation (6) is equivalent to

\[-\omega^2 + \omega_1^2 = \frac{\omega^2 (\omega^2 - \omega_0^2) k^4 (k^2 - \omega^2/c_P^2)}{c_s^2 (1 + \alpha/\mu) ((k^2 - \omega_1^2/c_P^2)^4 - k^6 (k^2 - \omega^2/c_P^2))}. \] (8)

Using this form of equation, we have investigated analytically three cases:
1) \( k \to \infty, \omega = O(1) \), 2) \( k \to \infty, \omega \to \infty \), and 3) \( k \to \omega_1/c_P, \omega \to \omega_1 \).
1) Case \( k \to \infty, \omega = O(1) \). From (8) we have

\[
\omega^2 - \omega_1^2 \to -\frac{\omega^2 - \omega_0^2}{(1 + \alpha/\mu)(2 - c_S^2/c_P^2)},
\]

which is equivalent to \( \omega \to \omega_2 = \omega_1 \sqrt{A/(1 + A)}, \quad A = 1 - \mu/(\lambda + 2\mu) > 1/2 \).

The boundary frequency \( \omega_2 \in (\omega_1/\sqrt{3}; \omega_1) \) for any elastic constants, it means that it always lies in the “allowed zone” \( \nu_1, \nu_2 \in \mathbb{R} \). We see the corresponding horizontal asymptote and the forbidden band above it in Figure 2.

2) Case \( k \to \infty, \omega \to \infty \). Looking for the solution \( \omega \approx ck \) in this limit, we obtain the cubic equation for \( \eta = c_S^2/c_P^2 \):

\[
\eta^3 - 8\eta^2 + 8(3 - 2pq)\eta - 16(2 - (p + q)) = 0,
\]

where \( p = c_S^2/c_P^2, q = \alpha/(\mu + \alpha) \). If \( \alpha = 0 \), this equation coincides with the classical case. In our case, if \( b > 0 \), the Rayleigh wave velocity in this limit will be the same as for the classical medium with larger \( c_P \). In general, \( b \in (-1; 1/2) \). Anyway, this equation has only one real (and positive) root, but \( c > c_S \) if \( b \leq -4/15 \), though \( c < 1.643c_S \) in any medium. It can be shown that \( c < c_S \alpha \) and \( c < c_P \). It means that this asymptote also lies in the “allowed zone” \( \nu_1, \nu_2 \in \mathbb{R} \) for any medium.

3) In the case \( \omega \to \omega_1, k \to \omega_1/c_P \) we may obtain from (8)

\[
k_1 = \frac{\omega_1}{c_P} \left( 1 + \frac{\omega - \omega_1}{\omega_1} \frac{\lambda^4(1 + \alpha/\mu)}{8\mu^2\alpha(\lambda + 2\mu)} \right).
\]

Only the upper part \( (\omega > \omega_1) \) of this curve lies in the “allowed zone”. We see that, indeed, in the numerical example the corresponding curve ends in the point under consideration.

There is a forbidden band of frequencies whose upper boundary is equal to \( \omega_1 \). One can prove (we omit the proof) that at \( \omega = \omega_1 - 0 \) we do not have solutions for the Rayleigh wave. It follows from this that either there is a forbidden zone lying under \( \omega_1 \) (as in the numerical example), or there an horizontal asymptote at \( \omega \to \omega_1 \) analogous to the case 2) for \( \omega_2 \). However, there is no possibility for existence of the horizontal asymptote at \( \omega_1 \), since the analysis gives that at \( k \to \infty, \omega = O(1) \) we have \( \omega = \omega_2 \). Thus we have a forbidden band of finite width below \( \omega_1 \). We have not obtained analytically the value of the lower boundary of the forbidden band. We can see from numerical results that at least for some parameters this is \( \omega_2 \), but we do not know if this is always so.
Conclusion

The rotational dynamics of inhomogeneities in rocks and soils taken into account by means of the reduced Cosserat continuum model gives a strongly dispersive behaviour for the Rayleigh wave in a certain range of frequencies. For some frequencies the Rayleigh wave does not propagate; for some wave numbers we have two possible frequencies of Rayleigh waves, and for some only one. The polarization is also different from the Rayleigh wave in the classical medium as well as in the Cosserat continuum with couple stresses. The difference manifests itself in the strongest way near the “forbidden band” of frequencies, where the energy of wave is trapped by the rotational dynamics of microstructure.

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References


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