21ST ICTAM 2004 WARSAW

INVESTIGATION OF COUPLE-STRESS EFFECTS IN ELASTIC BODIES UNDER DEFORMATION

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Vitold Nowacki

Theory of Elasticity (1970)

" …The model of the classical theory of elasticity agrees well with experiments conducted on construction materials at stresses within the limit of elasticity. Appreciable differences between the theory and experiment occur in the cases where stress gradients are essential …, in vibration problems of wave propagation and forced high-frequency vibrations ... and in granular materials…"

1887 the idea of the asymmetric theory of elasticity (W. Voigt)

1910 the first version of the general asymmetric elasticity theory (E.Cosserat, F. Cosserat)

60ies renewal and development of asymmetric elasticity theory (Trusdell, Toupin, Kuvshinskiy, Aero, Palmov, Eringen, Sukhubi, Nowacki, Morosov)

V. Nowacki **Theory of Elasticity (1970)**

“Future holds considerable promise for advancing general theory of the Cosserat continuum that, however, will require intensive experimental investigations and, in the first place, evaluation of all the material constants.”

Structure of the paper

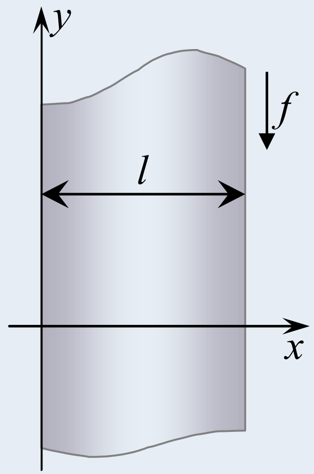
- new analytical solutions to problems of asymmetric elasticity theory;

- numerical solutions to problems of asymmetric elasticity theory;

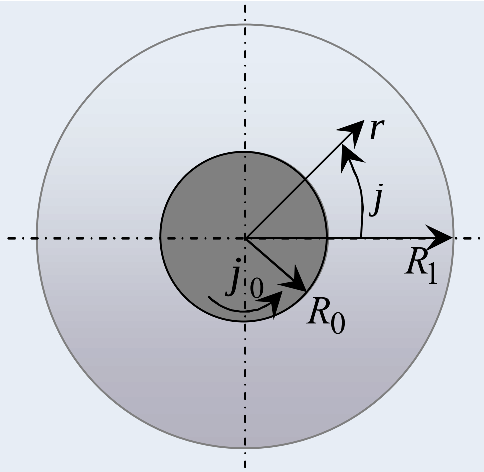
- analysis of the obtained solutions from the viewpoint of their utility for experimental developments;

- discussion of experimental results of couple-stress effects in the behavior of materials under deformation.

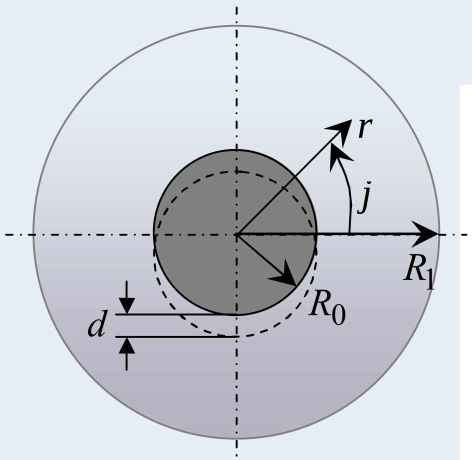
New analytical solutions to problems of the asymmetric elasticity theory



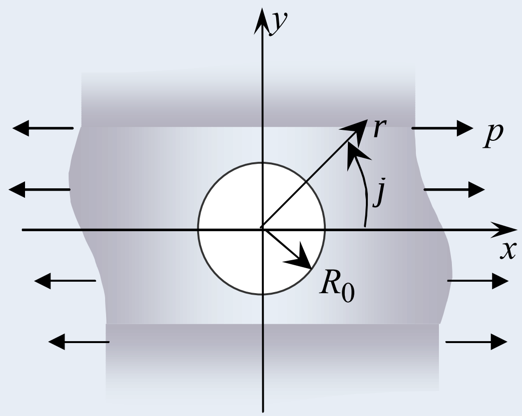
Deformation of an infinite plane layer fixed at both edges under the action of uniform mass forces



Deformation of a ring rigidly fixed along the outer contour due to rotation of the inner contour by a given angle

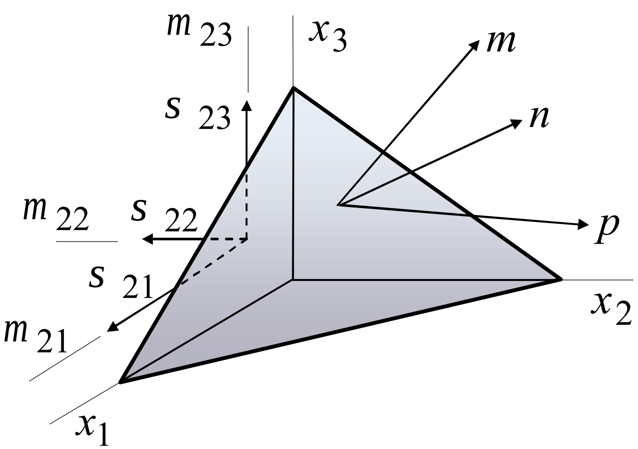


Deformation of a ring rigidly fixed along the outer contour due to displacement of the inner contour by a given value



The Kirsh problem of uniaxial stretching of an infinite plate with circular hole

Constitutive relations of the Cosserat continuum



s_{ij} - stress tensor

m_{ij} - couple-stress tensor

$\vec{p} = \vec{n} \cdot \tilde{s}, \quad \vec{m} = \vec{n} \cdot \tilde{m}$

$i \neq j \Rightarrow s_{ij} \neq s_{ji}$ - asymmetry

$$(2m + l)\,grad\;div\;\vec{u} - (m + a)\,rot\;rot\;\vec{u} + 2a\;rot\;\vec{w} + \vec{X} = 0$$

$$(2g + b)\,grad\;div\;\vec{w} - (g + e)\,rot\;rot\;\vec{w} + 2a\;rot\;\vec{u} - 4a\vec{w} + \vec{Y} = 0 \qquad (1)$$

Here:

$\vec{X}$ is the vector of mass forces;

$\vec{Y}$ is the vector of mass moments;

$\vec{u}$ is the displacement vector;

$\vec{w}$ is the rotation vector;

m, l are the Lame constants;

a, b, g, e are the physical constants of the material in the framework of the Cosserat continuum

Dimensionless parameters

$$A = l\sqrt{\frac{am}{(a+m)(g+e)}} \qquad B = \frac{a+m}{a} \qquad C = \frac{g-e}{g+e} \qquad (2)$$

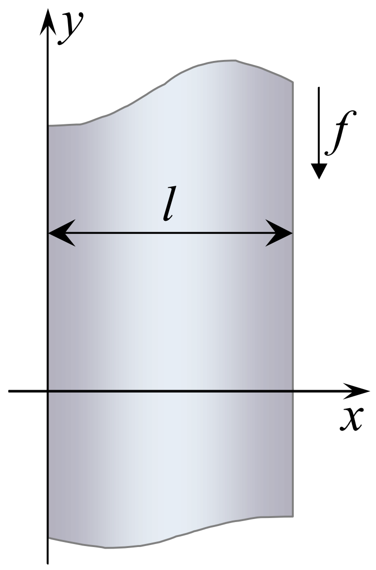
Inequalities for material constants (Ganthier)

$$3l + 2m + a \geq 0 \qquad 2m + a \geq 0 \qquad a \geq 0$$

$$3b + 2g \geq 0 \qquad |g - e| \leq g + e \qquad g + e \geq 0 \qquad (3)$$

$$A > 0 \qquad B \geq 1 \qquad |C| \leq 1 \qquad (4)$$

The form of solution



$$\vec{u}\big|_{x=l} = \vec{w}\big|_{x=l} = 0$$

$$\vec{u}\big|_{x=0} = \vec{w}\big|_{x=0} = 0$$

$$u_y(x) = C_1 + C_2 x + C_3 e^{2Ax} + C_4 e^{-2Ax} + \frac{f}{2} x^2$$

$$w_z(x) = \frac{C_2}{2} + C_3 AB e^{2Ax} - C_4 AB e^{-2Ax} + \frac{f}{2} x$$

(5)

$$g_{xy}(x) = \frac{C_2}{2} - C_3 A(B-2) e^{2Ax} + C_4 A(B-2) e^{-2Ax} + \frac{f}{2} x$$

$$g_{yx}(x) = \frac{C_2}{2} + C_3 AB e^{2Ax} - C_4 AB e^{-2Ax} + \frac{f}{2} x$$

$$c_{xz}(x) = 2C_3 A^2 B e^{2Ax} + 2C_4 A^2 B e^{-2Ax} + \frac{f}{2}$$

$$s_{yx}(x) = fx + C_2 + 4C_3 A e^{2Ax} - 4C_4 A e^{-2Ax} + \frac{f}{2} x$$

$$s_{xy}(x) = fx + C_2$$

$$m_{xz}(x) = 2C_3 e^{2Ax} + 2C_4 e^{-2Ax} + \frac{f}{2A^2 B}$$

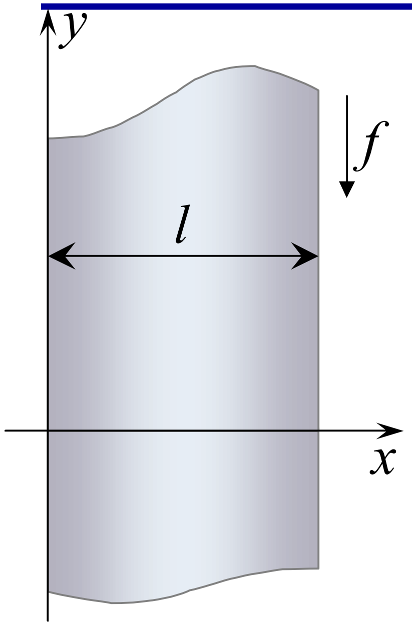
$$m_{zx}(x) = C\, m_{xz}(x)$$

M.A. Kulesh, V.P. Matveyenko, and I.N. Shardakov

Parametric analysis of analytical solutions to one- and two-dimensional problems in couple-stress theory of elasticity

ZAMM Z. Angew. Math. Mech. 83, No.4, 238-248 (2003)

Problem of layer deformation



R. Lakes:

$l = 2.096 \times 10^9$ н/м^2

$m = 1.033 \times 10^9$ н/м^2

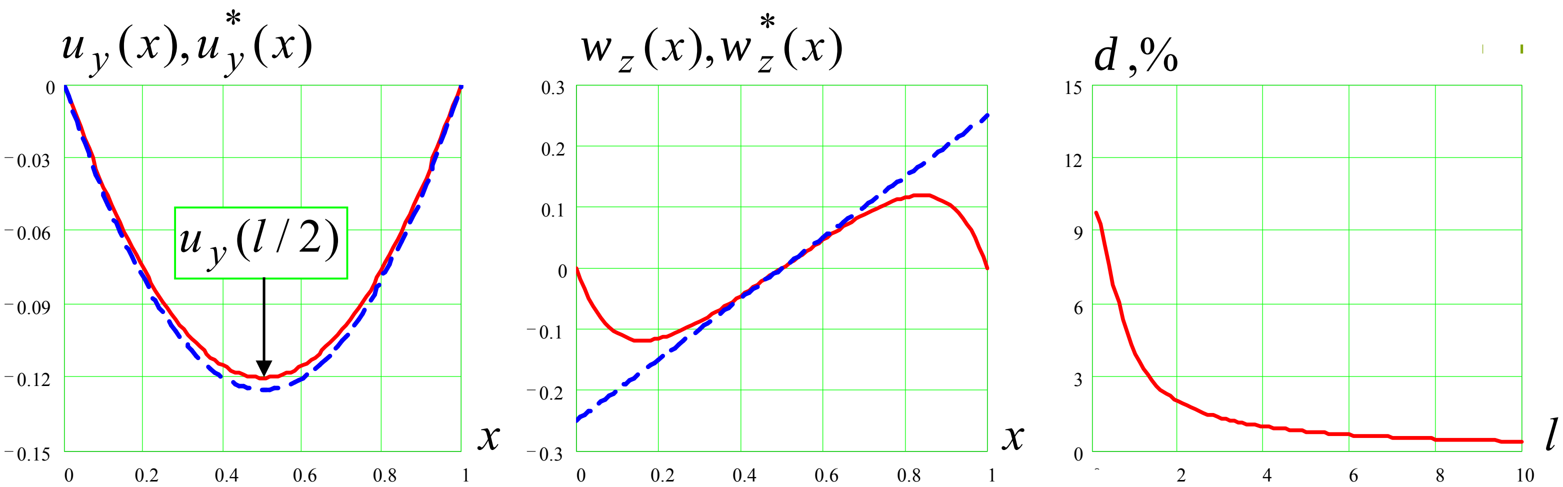
$a = 1.148 \times 10^8$ н/м^2

$n = 4.1 \times 10^6$ н

$e = 1.312 \times 10^5$ н

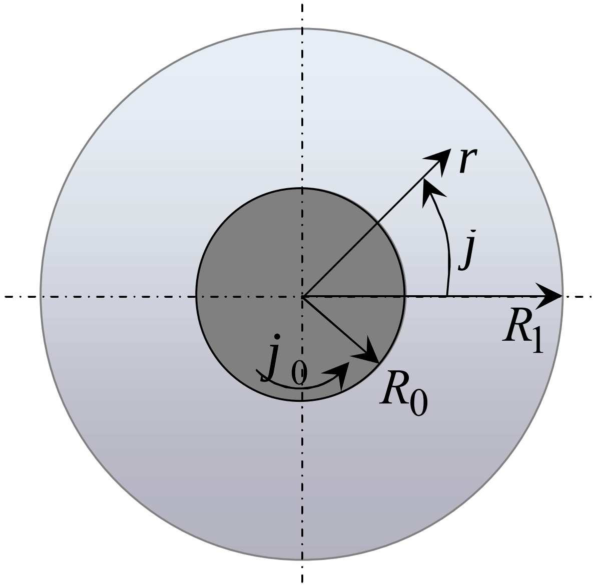
- couple-stress solution (solid red line)
- classical solution (dashed blue line)
- $u_y(l/2)$ – displacement in the middle of the layer
- d – degree of difference between the couple-stress (u_y) and classical (u_y^*) solutions

$$d = \left| \frac{u_y(l/2) - u_y^*(l/2)}{u_y^*(l/2)} \right| \times 100\%$$



Distribution of displacements and rotations throughout the layer thickness

Problem of ring torsion

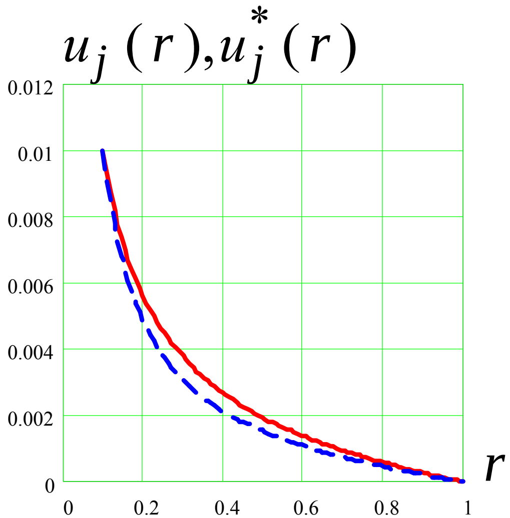


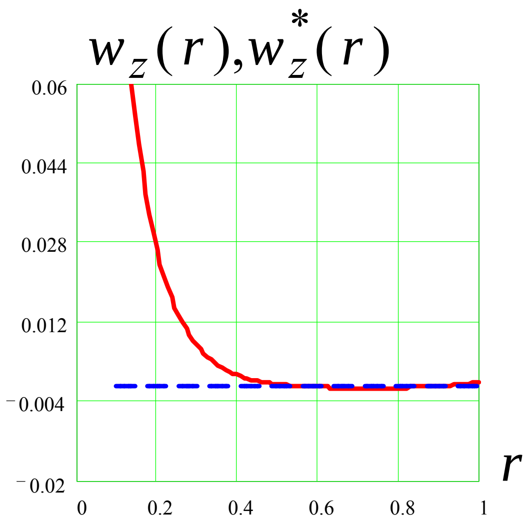
- couple-stress solution
- classical solution
- *M* – reactive torque

$$M = \int_0^{2p} \left[s_{rj}(R_0, j) R_0^2 + m_{rz}(R_0, j) R_0 \right] dj$$

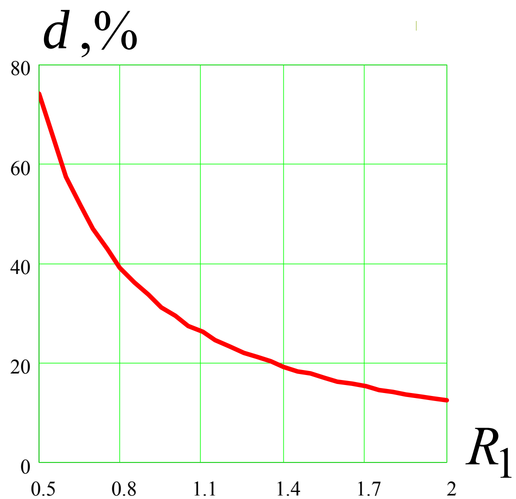
- *d* – degree of difference between the couple-stress *(M)* and classical *(M*)* solutions

$$d = \left| \frac{M - M^*}{M^*} \right| \times 100\,\%$$



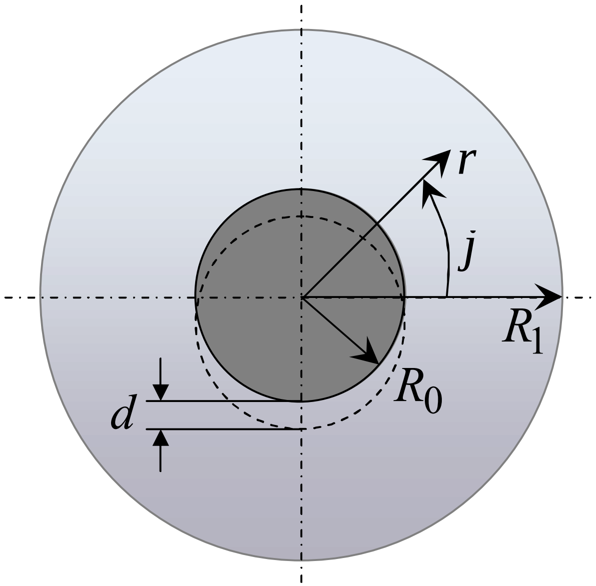


Distribution of displacements and rotations along the radius



$R_0 / R_1 = 0.1$

Problem of ring deformation

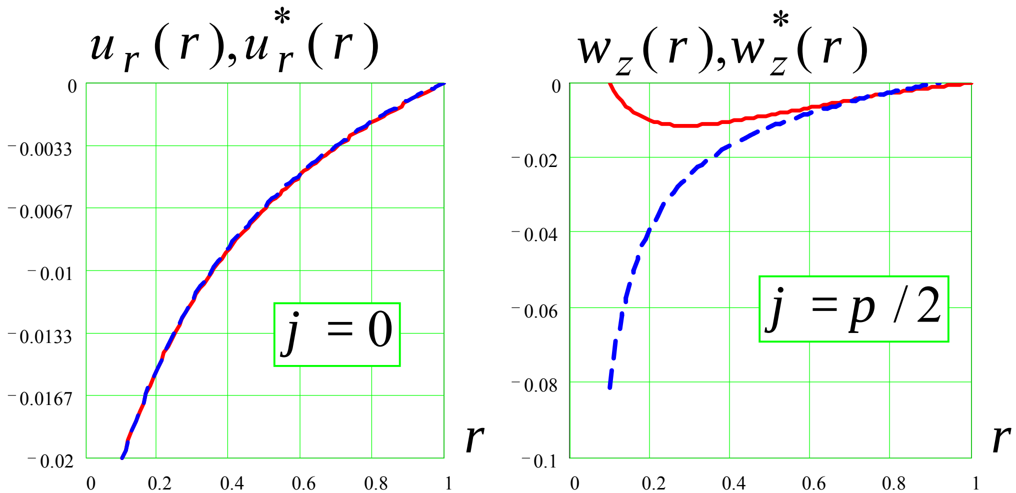


- couple-stress solution (solid red line)
- classical solution (dashed blue line)
- *F* – reactive force

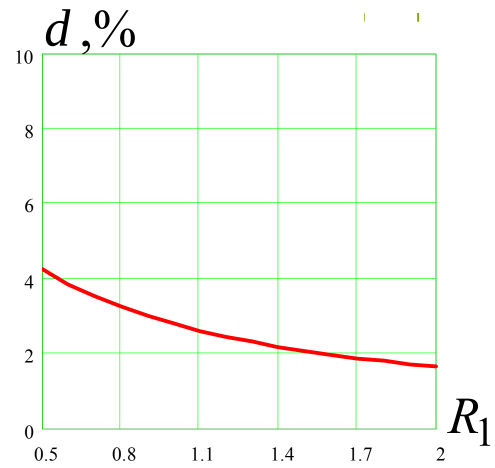
$$F = \int_0^{2\pi} \left[\sigma_{r\varphi}(R_0,\varphi)\sin\varphi + \sigma_{rr}(R_0,\varphi)\cos\varphi \right] d\varphi$$

- δ – degree of difference between the couple-stress *(F)* and classical solutions *(F*)*

$$\delta = \left| \frac{F_y - F_y^*}{F_y^*} \right| \times 100\,\%$$



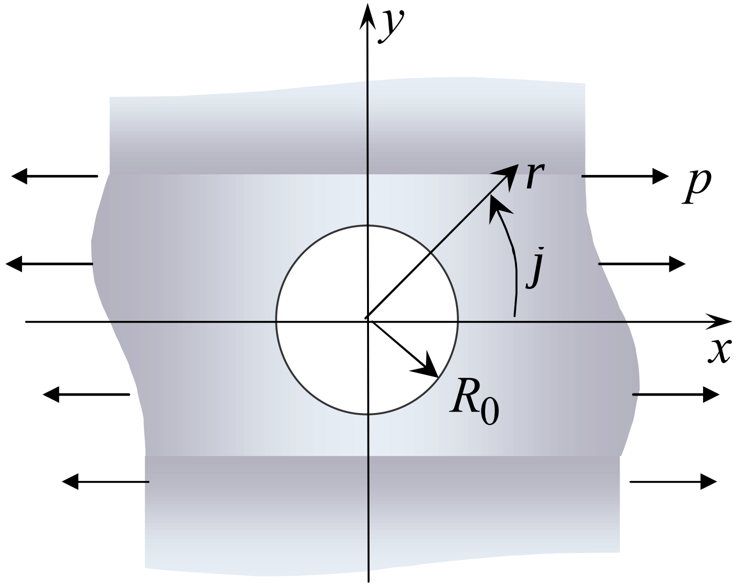
Distribution of displacements and rotations along the radius

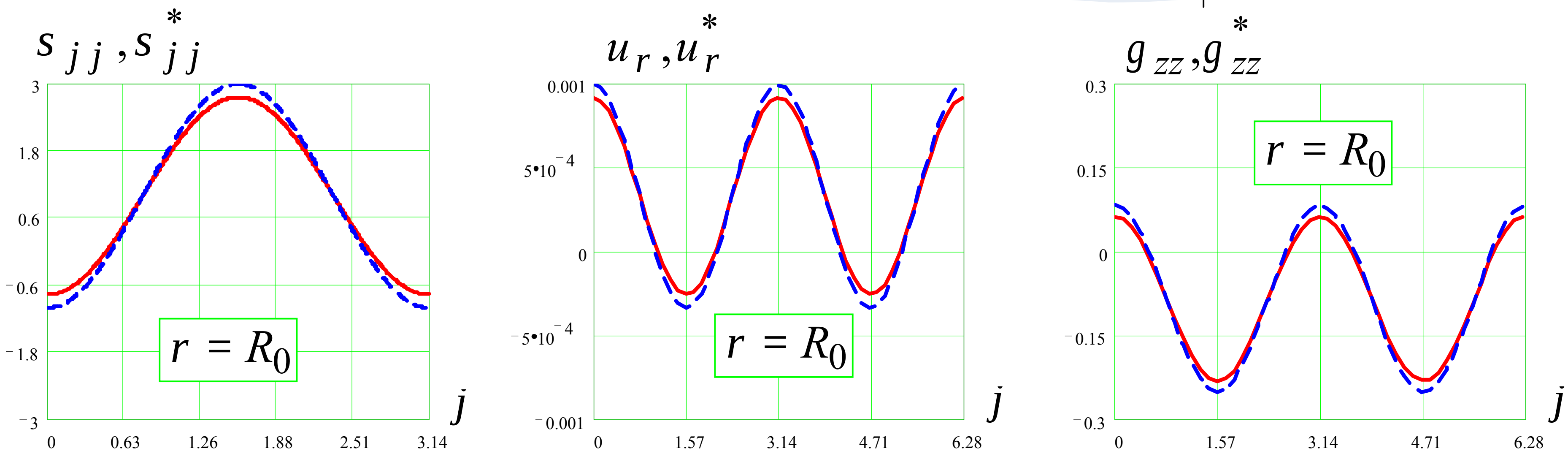


$R_0 / R_1 = 0.1$

Kirsh problem

- couple-stress solution ——
- classical solution - - - -





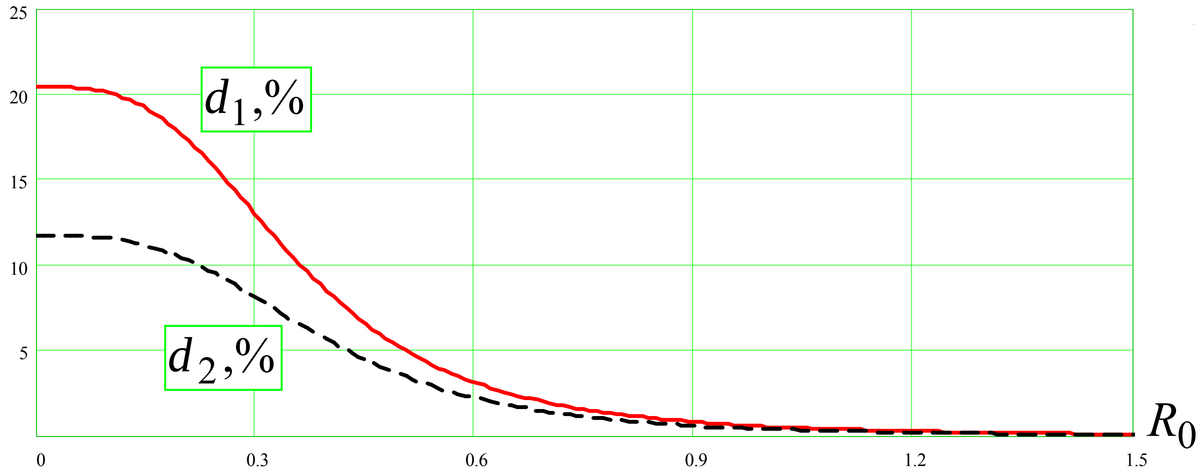
Distribution of stresses, displacements and strains along the inner contour ($R_0 = 0.01$)

Kirsh problem

$$D_1 = \left| \frac{u_r(R_0,0)}{u_r(R_0,p/2)} \right|$$

$$D_2 = \left| g_{zz}(R_0,0) - g_{zz}(R_0,p/2) \right|$$

In the classical theory D_1 is equal to 3



$$d_1 = \left| \frac{D_1 - D_1^*}{D_1^*} \right| \times 100\%$$

$$d_2 = \left| \frac{D_2 - D_2^*}{D_2^*} \right| \times 100\%$$

D_1, D_2 - couple-stress solution
D_1^*, D_2^* - classical solution

Finite-element realization of two-dimensional problems of asymmetric elasticity theory

Variational equation

$$\int_V \left(\tilde{\sigma} \cdot\cdot\, \delta\tilde{\gamma} + \tilde{\mu} \cdot\cdot\, \delta\tilde{\kappa}\right) dV - \int_V \left(\vec{X} \cdot \delta\vec{u} + \vec{Y} \cdot \delta\vec{w}\right) dV = \int_S \left(\vec{p} \cdot \delta\vec{u} + \vec{m} \cdot \delta\vec{w}\right) dS \qquad (6)$$

Here:

$\vec{X}$ is the vector of mass forces;

$\vec{Y}$ is the vector of mass moments;

$\vec{u}$ is the displacement vector;

$\vec{w}$ is the rotation vector;

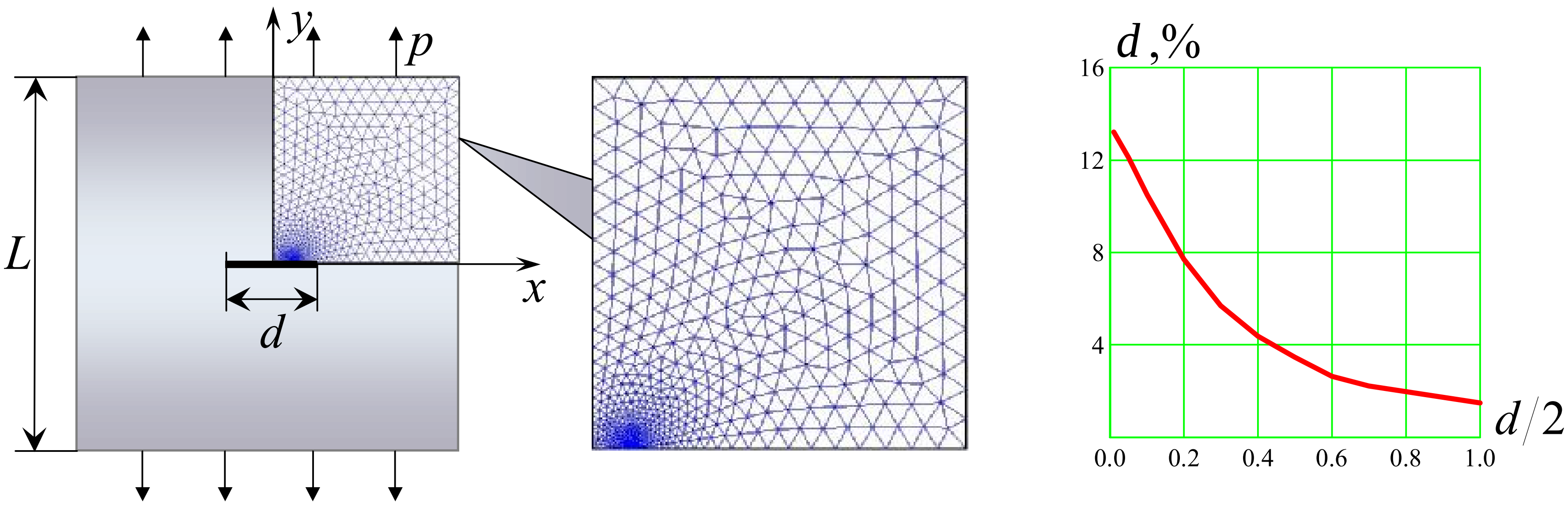
$\tilde{\gamma}$ and $\tilde{\kappa}$ are the deformation and bending-torsion tensors;

$\tilde{\sigma}$ and $\tilde{\mu}$ the stress and couple stress tensors;

$\vec{p}$ is the principal vector;

$\vec{m}$ is the principal moment

Problem of extension of a plate with a crack



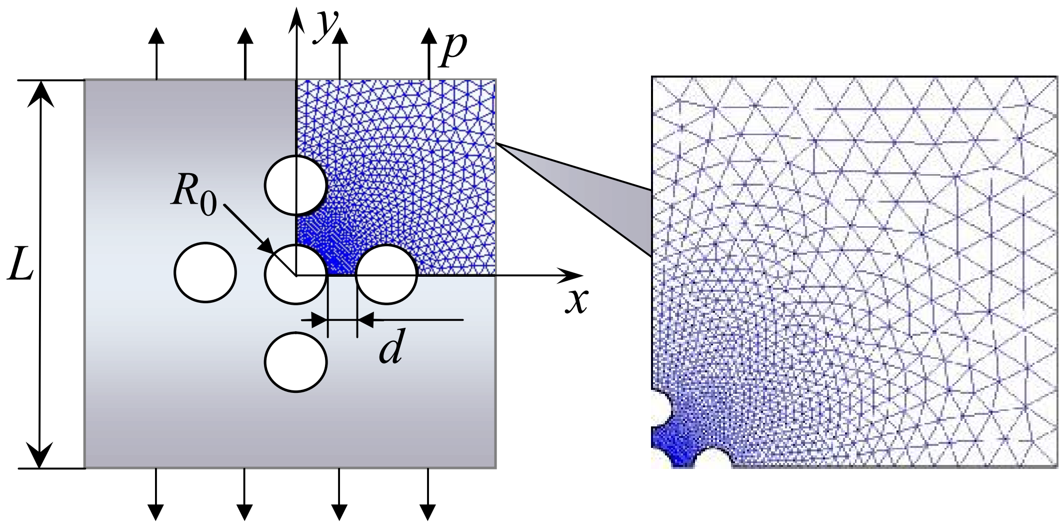
d - the degree of difference between the couple-stress and classical solutions

$$d = \left| \frac{u_y(0,0) - u_y^*(0,0)}{u_y^*(0,0)} \right| \cdot 100\,\%$$

u_y is the couple-stress solutions

u_y^* is the classical solutions

Problem of extension of a plate with five holes

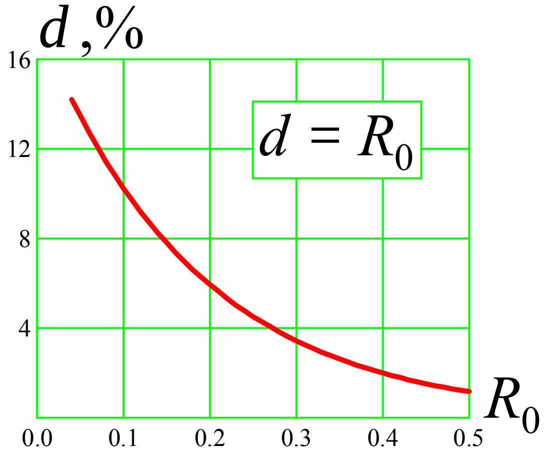


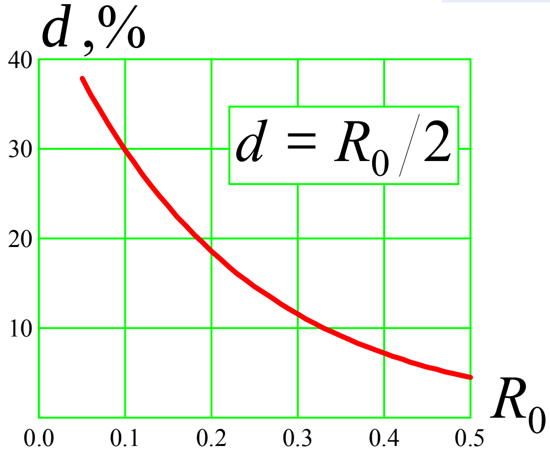
u_y is the couple-stress solutions

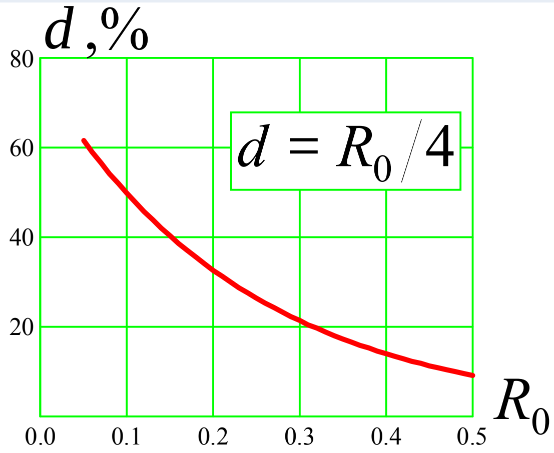
u_y^* is the classical solutions

d - the degree of difference between the couple-stress and classical solutions

$$d = \left| \frac{u_y(0,R_0) - u_y^*(0,R_0)}{u_y^*(0,R_0)} \right| \cdot 100\%$$







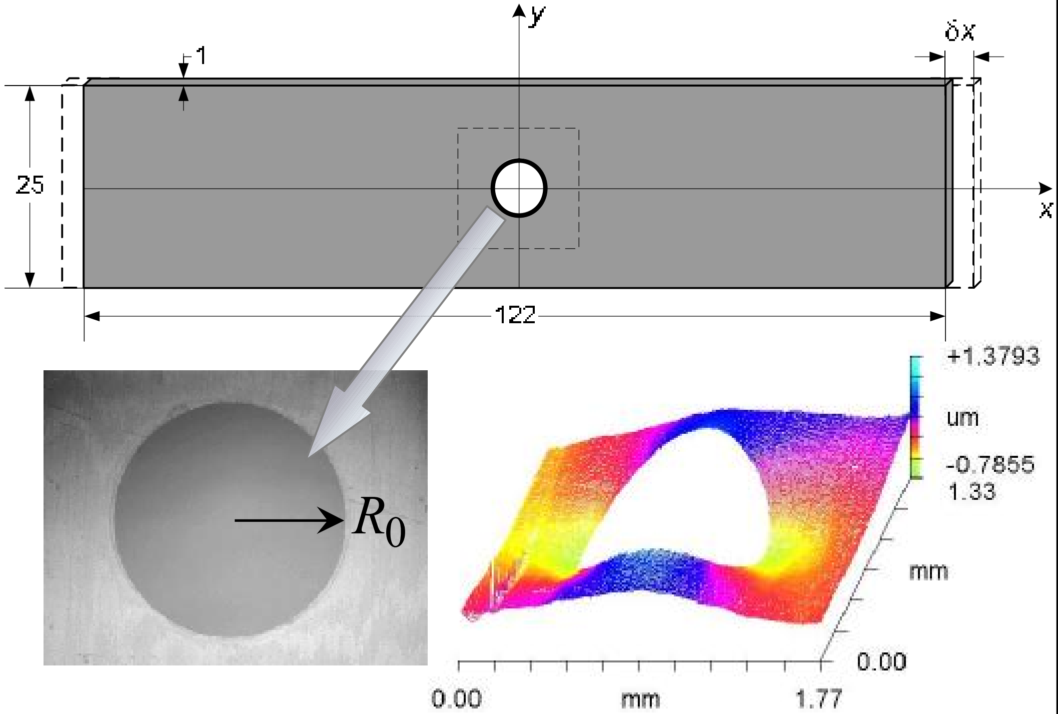
The degree of difference between the couple-stress and classical solutions at different values of the hole radius and inter-hole spacing

Experimental installation

Microscope-interferometer New View 5000

Loading unit

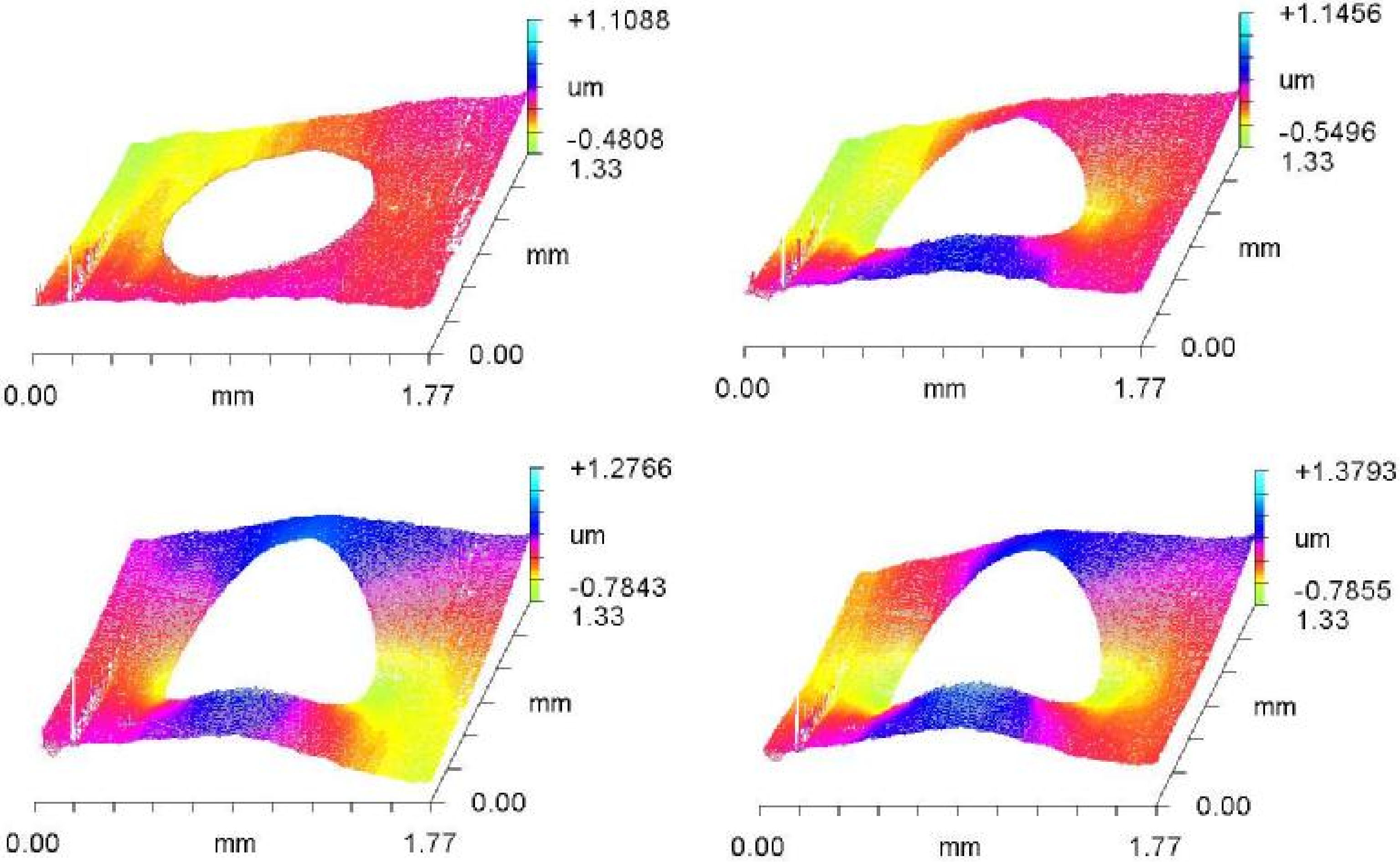
specimen material –
polymethylmethacrylate



module of elasticity – $E = 4,44 \cdot 10^8 \; H/m^2$

the Poisson ratio – $n = 0,33$

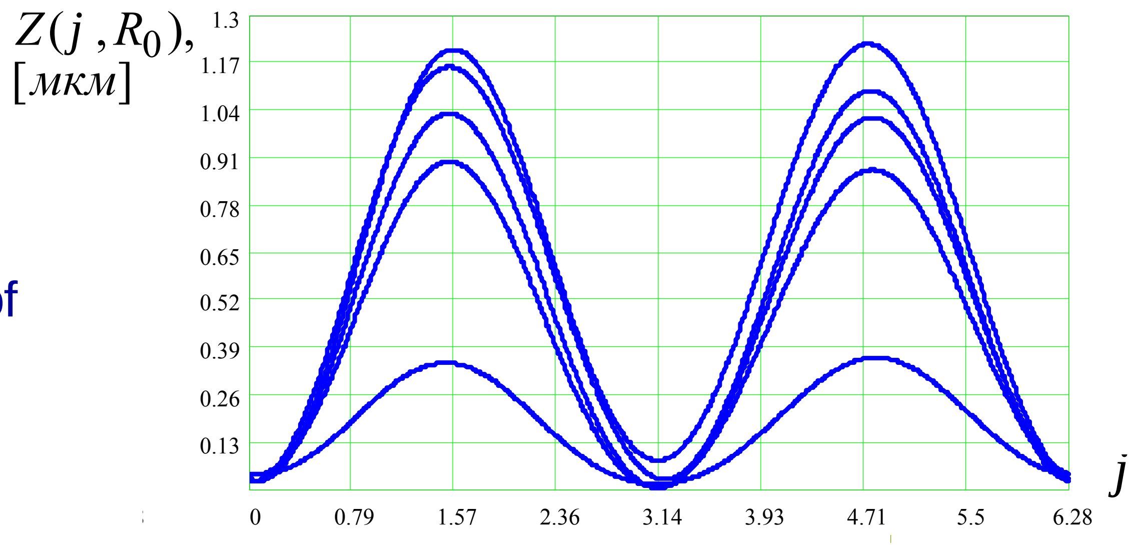
Experimental realization of the Kirsh problem



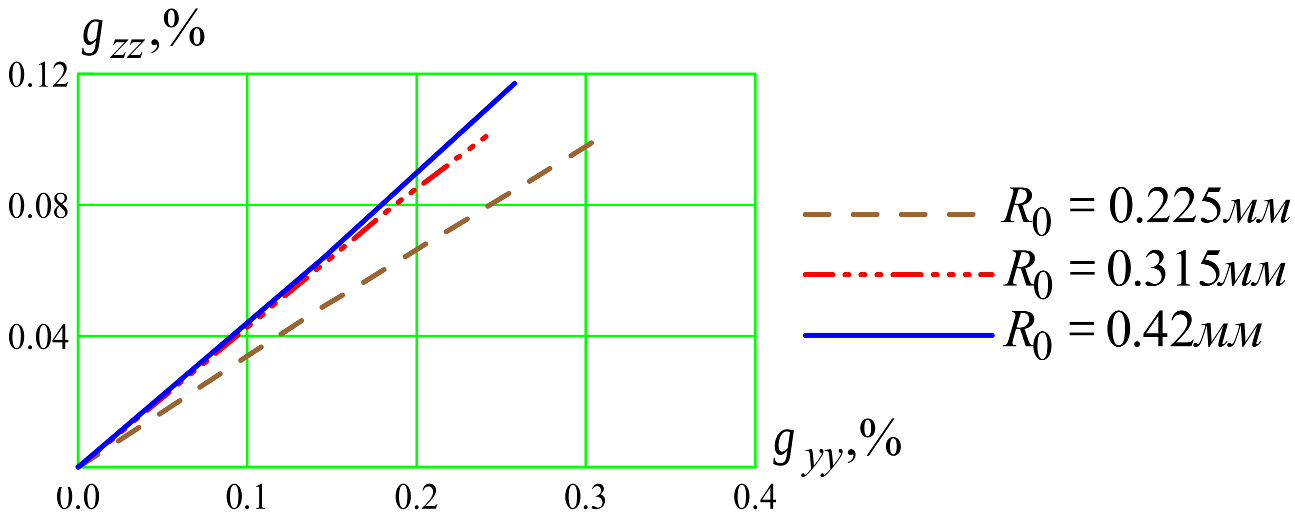
Experimental data on variation of the specimen surface profile at different loads

Experimental realization of the Kirsh problem

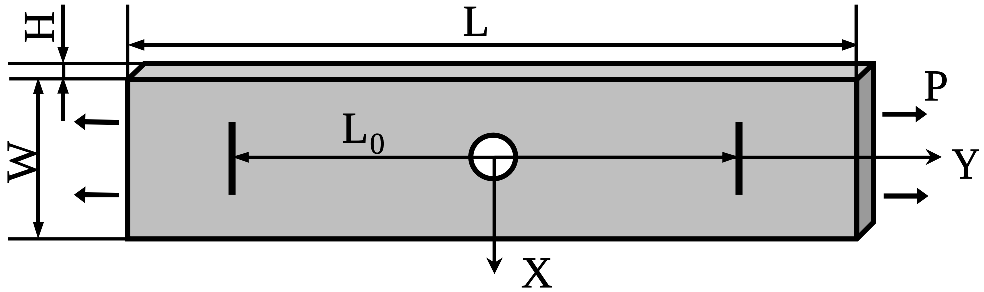
Variation of the surface contour along the hole contour at different degrees of specimen extension



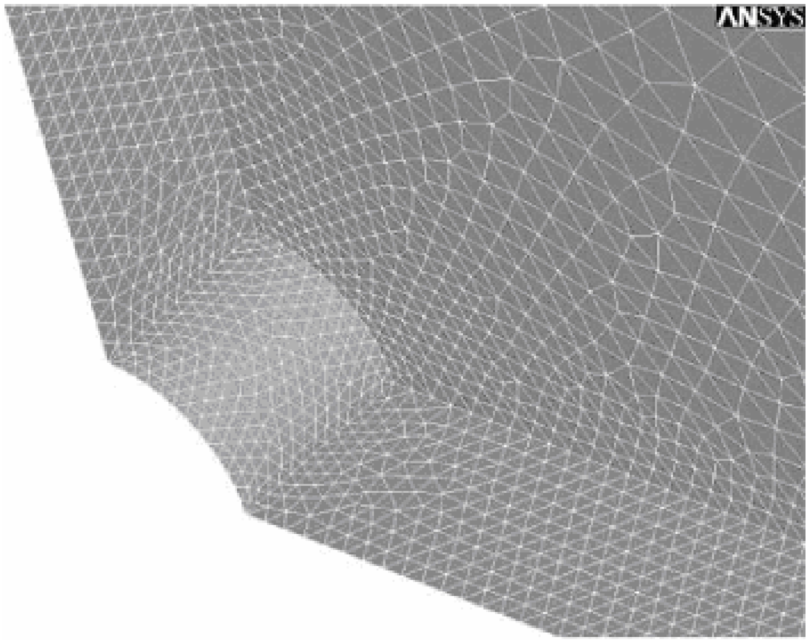
The plot of maximum strain g_{zz} (across the plate thickness) as a function of g_{yy}



3-D Problem on extension of a plate with a central hole



Discretization of computational domain for 3-D problem on extension of a plate with a hole



Conclusions

- Ø four new analytical solutions for the problems of asymmetric elasticity theory are obtained;
- Ø the obtained solutions are used to identify macroscopic parameters which can be measured by available methods in experimental realization of the corresponding problems;
- Ø a number of problems of asymmetric elasticity theory are simulated numerically to reveal the possibilities of their experimental implementation aimed at detection of the couple-stress effects in the behavior of materials;
- Ø comparison of the solutions obtained with their analogs from the classical elasticity theory is made to demonstrate the couple-stress effects;
- Ø experimental investigations are carried out to gain a deeper insight into the behavior of materials related to couple-stress effects in elastic bodies under deformation.

One of the goals of this talk is to provoke experimentalists to use the here – presented solutions for designing new experimental set-ups aimed at finding evidence of the couple-stress effects under deformations of elastic bodies. Closely related to that is the problem of developing new and refining the available schemes capable of identifying the elastic constants of the asymmetric elasticity theory.