MODELING OF WAVE DISPERSION USING CONTINUOUS WAVELET TRANSFORMS:
Incorporating Causality Constraint with Non-linear Frequency-dependent Attenuation

M. Holschneider[a], M. Kulesh[a], M. S. Diallo[b], K. Kurennaya[a], F. Scherbaum[c]

Institute of Mathematics[a], University of Potsdam, Am Neuen Palais 10, 14469 Potsdam, Germany
ExxonMobil Upstream Research Company[b], 3616 Richmond, Houston, USA
Institute of Geoscience[c], University of Potsdam, Karl-Liebknecht-Str. 24-25, 14415 Potsdam, Germany
Email: *mkulesh@math.uni-potsdam.de*
Internet: *http://www.math.uni-potsdam.de/~hols/DFG1114/projectseis.html*

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ABSTRACT

This contribution is concerned with the modeling of wave dispersion using continuous wavelet transforms. The mathematical basis for such modeling was formulated in *(M. Kulesh et al., Pure Appl. Geophys, 162(2005), p. 843-855)*. In that work we derived the wavelet propagator with the assumption that the attenuation shows a nearly linear frequency dependency without imposing any constraint on the relationship between the phase velocity and the attenuation. In the present work we show how to improve the model to incorporate non-linear frequency dependent attenuation for the Cauchy wavelet while satisfying the causality constraint. We show examples with synthetic data for illustration.

1 METHODOLOGY

- Definition of the wavelet transform of a signal $S \in L^2(\mathbb{R})$ is

$$\mathcal{W}_g S(t,a) = \int_{-\infty}^{+\infty} \frac{1}{a} g^*\left(\frac{\tau - t}{a}\right) S(\tau)\, d\tau,$$

where $g(\tau)$ is the mother wavelet, a is the scale parameter (dilation), $t \in \mathbb{R}$ is the location parameter (translations) and $(\cdot)^*$ indicates the complex conjugate.

- The wavelet transform can be expressed in terms of the Fourier transform $\hat{S}(\zeta)$ of the signal $S(t)$,

$$\mathcal{W}_g S(t,a) = \int_{-\infty}^{+\infty} \hat{g}^*(a\zeta) e^{2\pi i t\zeta} \hat{S}(\zeta)\, d\zeta,$$

where $\hat{g}(.)$ is the Fourie transform of the mother wavelet.

- The signal $S(t)$ can be recovered from its wavelet transform:

$$S(t) = \frac{1}{C_{g,m}} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{a^2} m\left(\frac{t-\tau}{a}\right) \mathcal{W}_g S(\tau, a)\, d\tau\, da,$$

$$C_{g,m} = \int_{0}^{+\infty} \left(\hat{g}^*(\omega)\hat{m}(\omega) + \hat{g}^*(-\omega)\hat{m}(-\omega)\right) \frac{d\omega}{\omega},$$

where $m(t)$ is synthesizing wavelet, and ω is the angular frequency.

- Scale parameter a can be related to the physical frequency f by $a = f_0/f$.
- If $S_i(t)$ and $S_j(t)$ represent two signals observed at two stations, a distance D apart, the relation between the Fourier transforms of the two signals reads

$$\hat{S}_j(f) = e^{-i\mathbb{K}(f)D} \hat{S}_i(f),$$

where $\mathbb{K}(f)$ is the complex wavenumber which defined by frequency-depended real wavenumber $k(f)$ and attenuation $\alpha(f)$:

$$\mathbb{K}(f) = 2\pi k(f) - i\alpha(f).$$

2 ASYMPTOTIC PROPAGATOR IN WAVELET SPACE

Let us assume that the attenuation shows a nearly linear frequency dependence. In such case $\alpha'(f) = 0$ and the asymptotic propagator in wavelet space *(M. Kulesh et al., Pure Appl. Geophys, 162(2005), p. 843-855)* is given by:

$$\mathcal{W}_g S_j(t,f) = e^{-\alpha(f)D} e^{-2\pi i[k(f) - f k'(f)]D} \mathcal{W}_g S_i\left(t - k'(f)D, f\right).$$

- In the special case, with the assumption that the analyzing wavelet has linear phase (with time-derivative approximately equal to 2π, as it is the case for the Morlet wavelet), the approximation can be written in terms of the phase $V_p(f)$ and group $V_g(f)$ velocities as:

$$\mathcal{W}_g S_j(t,f) = \\ = e^{-\alpha(f)D} \left|\mathcal{W}_g S_i\left(t - \tfrac{D}{V_g(f)}, f\right)\right| \exp\left[i \arg \mathcal{W}_g S_i\left(t - \tfrac{D}{V_p(f)}, f\right)\right].$$

- Extra it is possible to express the cross-correlation of any chosen pair of traces in terms of the autocorrelation of a selected reference trace in the wavelet domain *(M. Holschneider, M. S. Diallo, M. Kulesh et al., Geophys. J. Int. 163(2005), p. 463-478)*

$$\mathcal{W}_g T_{ij}(t,f) = \\ = e^{-\alpha(f)\Delta_{ij}} e^{-2\pi i[k(f) - f k'(f)]D_{ij}} \mathcal{W}_g T_{rr}\left(t - k'(f)D_{ij}, f\right),$$

where the distance matrices are

$$\Delta_{ij} = D_i + D_j - 2D_r \quad \text{and} \\ D_{ij} = D_i - D_j.$$

Note: the frequency-depended wavenumber and attenuation are independent and therefore do not satisfy the causality constraint. In the next section we improve the wavelet progator model to satisfy this constraint.

3 ASYMPTOTIC PROPAGATOR IN WAVELET SPACE INCLUDING CAUSALITY CONSTRAINT

The challenge is to express the spectral propagator in terms of the wavelet transform of the source signal, $\mathcal{W}_g S_i(t,a)$ and propagated signal, $\mathcal{W}_g S_j(t,a)$ using complex wavenumber:

$$\mathcal{W}_g S_j(t,f) = \int_{-\infty}^{+\infty} \hat{g}^*(\zeta/f) e^{2\pi i t\zeta} e^{-i\mathbb{K}(f)D} \hat{S}_i(\zeta)\, d\zeta.$$

- Assumption: frequency-depended wavenumber and attenuation are slowly varying with respect to the frequency range of the mother wavelet. For moderate dispersion, complex wavenumber can be approximated by the first two terms of its Taylor series around frequency f:

$$\mathbb{K}(\zeta) = \mathbb{K}(f) + (\zeta - f)\mathbb{K}'(f) + O(|\zeta - f|^2).$$

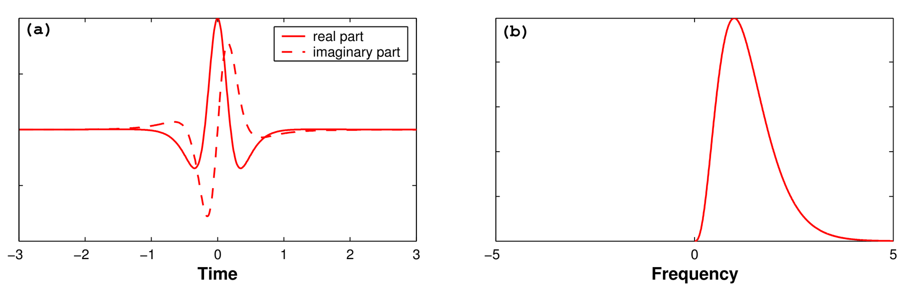
Upon inserting the above approximations into the integral we obtain

$$\mathcal{W}_g S_j(t,f) = e^{-i[\mathbb{K}(f) - f\mathbb{K}'(f)]D} \int_{-\infty}^{+\infty} \hat{g}^*(\zeta/f) e^{2\pi i[t - \mathbb{K}'(f)D/(2\pi)]\zeta} \hat{S}_i(\zeta)\, d\zeta.$$

- Complex Cauchy wavelet of power p

$$g(t) = \left(1 - \frac{2\pi i t}{p-1}\right)^{-p}, \qquad \hat{g}(f) = f^{p-1} e^{-f(p-1)}.$$

Figure 1: Complex Cauchy wavelet in the time and the freqency domain



- Using the complex Cauchy wavelet we can absorb the imagenery part of time shift concerned with the attenuation:

$$\hat{g}^*(\zeta/f) e^{\Im\mathbb{K}'(f)D\zeta} = \frac{\hat{g}^*(f_\alpha(f)\zeta/f)}{f_\alpha^{p-1}(f)},$$

where

$$f_\alpha(f) = 1 - \frac{fD}{p-1}\Im\mathbb{K}'(f).$$

- Using this property of Cauchy wavelet we derive the asymptotic propagator in wavelet space:

$$\mathcal{W}_g S_j(t,f) = \frac{1}{f_\alpha^{p-1}(f)} e^{-i[\mathbb{K}(f) - f\mathbb{K}'(f)]D} \mathcal{W}_g S_i\left(t - \tfrac{D}{2\pi}\Re\mathbb{K}'(f), \tfrac{f}{f_\alpha(f)}\right).$$

4 ASYMPTOTIC CROSS-CORRELATION PROPAGATOR IN WAVELET SPACE

- Definitions. We consider a set of seismic data consisting of N traces $S_i(t)$ from N aligned stations.
  - The autocorrelation of the reference signal $S_r(t)$ at a distance D_r is

$$T_{rr}(t) = S_r(t) \otimes S_r(t), \quad \hat{T}_{rr}(f) = |\hat{S}_r(f)|^2$$

  - The cross-correlation of $S_i(t)$ at a distance D_i and $S_j(t)$ at a distance D_j is

$$T_{ij}(t) = S_i(t) \otimes S_j(t), \quad \hat{T}_{ij}(f) = \hat{S}_i(f) \cdot \hat{S}_j^*(f)$$

- Fourier propagator. The relation between the Fourier transforms of cross-correlation and autocorrelation is

$$\hat{T}_{ij}(f) = e^{-i[\mathbb{K}(f)D_{ir} - \mathbb{K}^*(f)D_{jr}]} \hat{T}_{rr}(f)$$

- Wavelet propagator. By analogy to the derivation of asymptotic propagator in wavelet-space, we can express the wavelet transform of T_{ij} in terms of the wavelet transform of T_{rr} as follows

$$\mathcal{W}_g T_{ij}(t,f) = \\ = \frac{e^{-i[(\mathbb{K}(f) - f\mathbb{K}'(f))D_{ir} - (\mathbb{K}^*(f) - f\mathbb{K}^{*\prime}(f))D_{jr}]}}{f_\alpha^{p-1}(f)} \mathcal{W}_g T_{rr}\left(t - \tfrac{D_{ij}}{2\pi}\Re\mathbb{K}'(f), \tfrac{f}{f_\alpha(f)}\right),$$

where

$$f_\alpha(f) = 1 - \frac{f\Delta_{ij}}{p-1}\Im\mathbb{K}'(f).$$

Note: all operator forms given in this work are equivalent in the sense that they can all be used to model the propagation of a source wavelet in the medium when the attenuation and dispersion characteristics are known. The advantage of using a wavelet propagator derived from the cross-correlation becomes more perceptible when dealing with the inverse problem, namely, the attempt to estimate the dispersion and attenuation characteristics from recorded signals in an actual seismic survey.

5 AN EXAMPLE: APPLICATION TO THE COLE-COLE MODEL

For example let us consider an isotropic linear viscoelastic mudium, which can be described using Cole-Cole model *(Jian-Fei Lu and Andrzej Hangya, Numerical modelling method for wave propagation in a linear viscoelastic medium with singular memory, Geophys. J. Int. 159(2004), p. 688-702)*:

- The stress-strain constitutive relation of the Cole-Cole model is determinated by complex modulus

$$M(\omega) = M_r \frac{1 + (i\omega\tau_\varepsilon)^\gamma}{1 + (i\omega\tau_\sigma)^\gamma}.$$

- The complex wavenumber are given by

$$\mathbb{K}(\omega) = \frac{\omega}{\sqrt{M(\omega)}}.$$

- The frequency-dependent attenuation and dispersion are written as

$$k(f) = \Re\mathbb{K}(2\pi f)/(2\pi), \qquad \alpha(f) = -\Im\mathbb{K}(2\pi f).$$

Figure 2: (a) Phase and group velocities and (b) attenuation curve for Cole-Cole model with parameters: $M_r = 7.87 \cdot 10^6$, $\gamma = 0.4$, $\tau_\varepsilon = 4.73 \cdot 10^{-4}$, $\tau_\sigma = 1.71 \cdot 10^{-4}$.

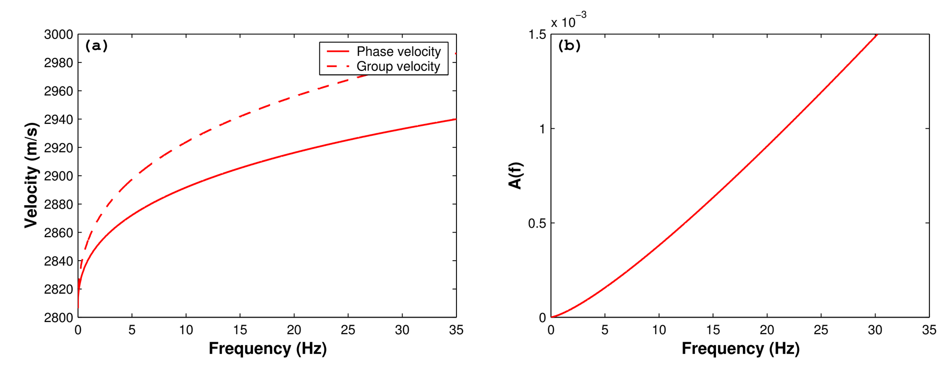


Figure 3: Synthetic example: (a) propagated Shanon wavelet using Fourier method (red) and the wavelet operator (black dashed line), (b) modulus of the wavelet transforms of the different waveforms and (c) the corresponding phase pictures.

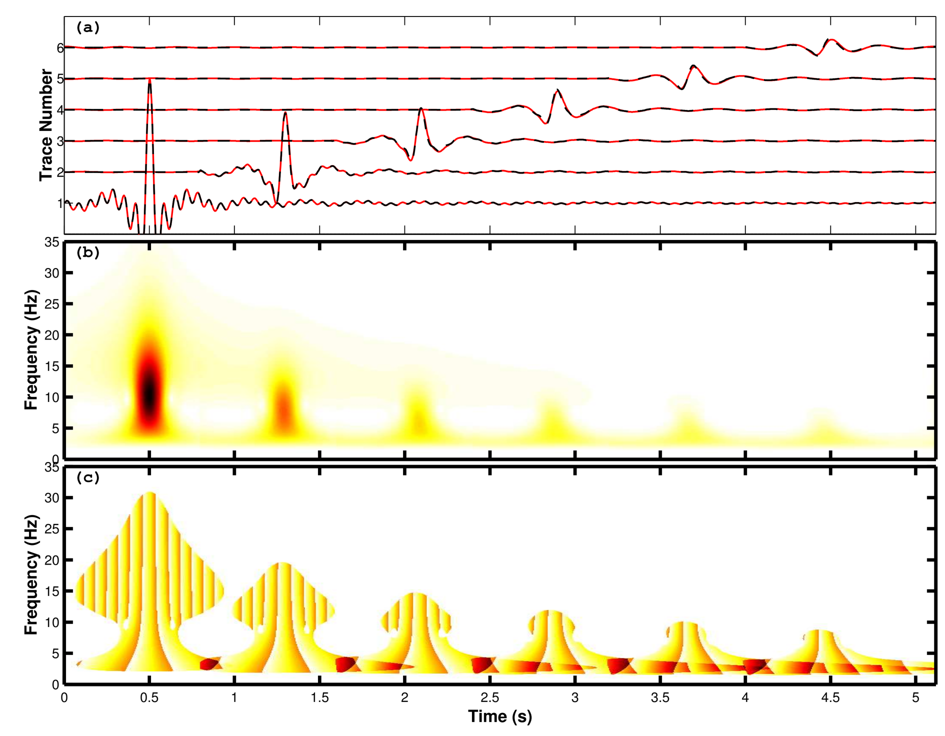
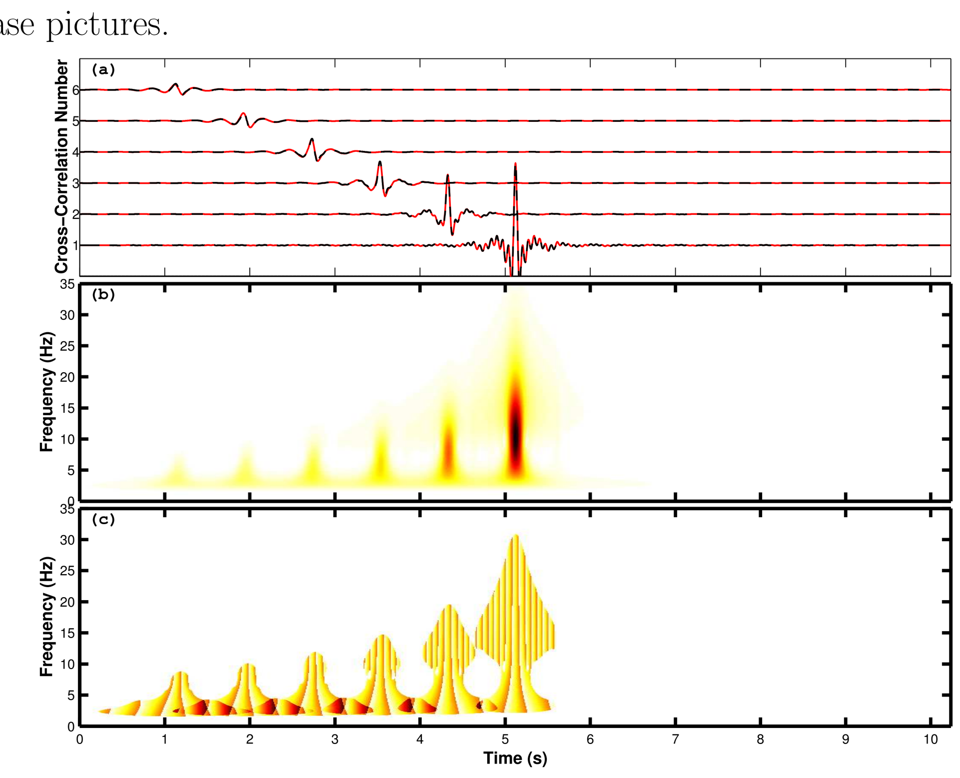


Figure 4: Synthetic example: (a) propagated cross-correlation using Fourier method (red) and the wavelet operator (black dashed line), (b) modulus of the wavelet transforms of the different waveforms and (c) the corresponding phase pictures.



CONCLUSION

- An operator given the relation between the wavelet spectrums of signals from aligned stations is considered. We assume that the wave propagates in medium with dispersion and attenuation.
- We expand this operator using complex Cauchy wavelet in case when the relationship between the phase velocity and the attenuation is satistfied by causality constraint.
- We also considered a set of seismic data consisting of N traces from N aligned stations. In such case we adapted our operator for the description of the relation between autocorrelation of a source signal and cross-correlation of two any signals in wavelet domain.

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