SURFACE WAVES PROPAGATION IN COSSERAT CONTINUUM
Construction of Solution and Analysis Using Wavelet Transform

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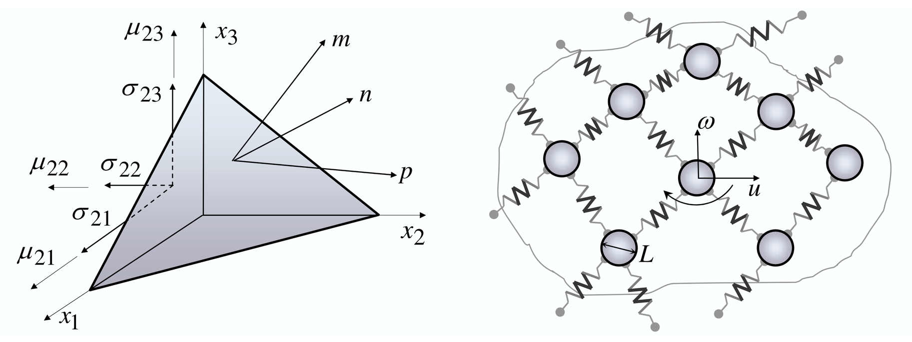
ABSTRACT

In this contribution we consider a problem of the surface elastic waves propagation in a half-space (Rayleigh wave) and in a thin layer (Lamb wave) within the framework of the Cosserat continuum. The medium deformation in this model is described not only by the displacement vector, but also by kinematically independent rotation vector. We obtained the general solution of equations of motion. This solution describes four wave types: Rayleigh wave and surface transverse wave in a half-space as well as Lamb wave and transverse wave in a thin layer. Our solutions show that within the framework of Cosserat continuum, both the Rayleigh and surface transverse waves in a half-space are dispersive. The transverse wave in a thin layer and surface transverse wave in a half-space are new wave modes and do not have any analogies in the classical elasticity theory.

1 ASYMMETRIC THEORY OF ELASTICITY

Figure 1: Definition of rotation vector and couple-stress tensor



Basic relations of the elastic Cosserat medium *(W. Nowacki. Teoria niesymetrycznej sprezystosci. Warszawa, PNW (1971))*:

- Force and moment of force $\boldsymbol{p} = \boldsymbol{n}\cdot\tilde{\sigma}$, $\boldsymbol{m} = \boldsymbol{n}\cdot\tilde{\mu}$
- Displacement vector $\boldsymbol{u} = \{u_x, u_y, u_z\}$
- Rotation vector $\boldsymbol{\omega} = \{\omega_x, \omega_y, \omega_z\}$
- Asymmetric strain tensor $\tilde{\gamma} = \nabla\boldsymbol{u} - \tilde{\boldsymbol{E}}\cdot\boldsymbol{\omega}$
- Asymmetric torsion bending tensor $\tilde{\chi} = \nabla\boldsymbol{\omega}$
- Asymmetric stress tensor $\tilde{\sigma} = 2\mu\tilde{\gamma}^{(S)} + 2\alpha\tilde{\gamma}^{(A)} + \lambda I_1(\tilde{\gamma})\tilde{e}$
- Asymmetric couple-stress tensor $\tilde{\mu} = 2\gamma\tilde{\chi}^{(S)} + 2\varepsilon\tilde{\chi}^{(A)} + \beta I_1(\tilde{\chi})\tilde{e}$
- Equilibrium equations

$$\nabla\cdot\tilde{\sigma} + \boldsymbol{X} = \rho\ddot{\boldsymbol{u}}, \quad \tilde{\sigma}^T : \tilde{\boldsymbol{E}} + \nabla\cdot\tilde{\mu} + \boldsymbol{Y} = j\ddot{\boldsymbol{\omega}}$$

- Equations of motion

$$(\lambda+2\mu)\operatorname{grad}\operatorname{div}\boldsymbol{u} - (\mu+\alpha)\operatorname{rot}\operatorname{rot}\boldsymbol{u} + 2\alpha\operatorname{rot}\boldsymbol{\omega} + \boldsymbol{X} = \rho\ddot{\boldsymbol{u}},$$
$$(\beta+2\gamma)\operatorname{grad}\operatorname{div}\boldsymbol{\omega} - (\gamma+\varepsilon)\operatorname{rot}\operatorname{rot}\boldsymbol{\omega} + 2\alpha\operatorname{rot}\boldsymbol{u} - 4\alpha\boldsymbol{\omega} + \boldsymbol{Y} = j\ddot{\boldsymbol{\omega}},$$

where μ and λ are the Lame constants, α, β, γ and ε are the physical constants of a material in the framework of the Cosserat medium, ρ is the density, j is the moment of inertia density, $(.)^{(S)}$ and $(.)^{(A)}$ denote the symmetric and antisymmetric parts of tensor and $\tilde{\boldsymbol{E}}$ is the Levi-Civita tensor of the third rank.

2 THE GENERAL SOLUTION

- Representation of the solution for displacement and rotation vectors components

$$u_n(x,z,t) = \int_{-\infty}^{\infty} U_n(z)e^{i(kx+ft)}\hat{u}_0(f)\,df,$$
$$\omega_n(x,z,t) = \int_{-\infty}^{\infty} W_n(z)e^{i(kx+ft)}\hat{u}_0(f)\,df,$$

where k is the wavenumber, f is the circular frequency, t is the time, $U_n(z)$ and $W_n(z)$ are amplitude functions depending on depth and frequency. $\hat{u}_0(f)$ is the complex spectral function corresponding to the Fourier spectrum of a source signal and determines the wavepacket form.

- Displacement vector components

$$u_x(x,z,t) = \int_{-\infty}^{\infty}\Big\{D_1 ike^{-\nu_1 z} + D_2\nu_2 e^{-\nu_2 z} + D_3\nu_3 e^{-\nu_3 z} + D_4 ike^{\nu_1 z} - D_5\nu_2 e^{\nu_2 z} - D_6\nu_3 e^{\nu_3 z}\Big\}e^{i(kx+ft)}\hat{u}_0(f)\,df$$

$$u_y(x,z,t) = \frac{F}{2}\int_{-\infty}^{\infty}\Big\{E_2\Big(A_m - \frac{f^2}{C_4^2} + \frac{4}{F}\Big)e^{-\xi_2 z} + E_3\Big(A_p - \frac{f^2}{C_4^2} + \frac{4}{F}\Big)e^{-\xi_3 z} + E_5\Big(A_m - \frac{f^2}{C_4^2} + \frac{4}{F}\Big)e^{\xi_2 z} + E_6\Big(A_p - \frac{f^2}{C_4^2} + \frac{4}{F}\Big)e^{\xi_3 z}\Big\}e^{i(kx+ft)}\hat{u}_0(f)\,df$$

$$u_z(x,z,t) = \int_{-\infty}^{\infty}\Big\{-D_1\nu_1 e^{-\nu_1 z} + D_2 ike^{-\nu_2 z} + D_3 ike^{-\nu_3 z} + D_4\nu_1 e^{\nu_1 z} + D_5 ike^{\nu_2 z} + D_6 ike^{\nu_3 z}\Big\}e^{i(kx+ft)}\hat{u}_0(f)\,df$$

- Rotation vector components

$$\omega_x(x,z,t) = \int_{-\infty}^{\infty}\Big\{E_1 ike^{-\xi_1 z} + E_2\xi_2 e^{-\xi_2 z} + E_3\xi_3 e^{-\xi_3 z} + E_4 ike^{\xi_1 z} - E_5\xi_2 e^{\xi_2 z} - E_6\xi_3 e^{\xi_3 z}\Big\}e^{i(kx+ft)}\hat{u}_0(f)\,df$$

$$\omega_y(x,z,t) = \frac{B}{2}\int_{-\infty}^{\infty}\Big\{D_2\Big(A_m - \frac{f^2}{C_3^2}\Big)e^{-\nu_2 z} + D_3\Big(A_p - \frac{f^2}{C_3^2}\Big)e^{-\nu_3 z} + D_5\Big(A_m - \frac{f^2}{C_3^2}\Big)e^{\nu_2 z} + D_6\Big(A_p - \frac{f^2}{C_3^2}\Big)e^{\nu_3 z}\Big\}e^{i(kx+ft)}\hat{u}_0(f)\,df$$

$$\omega_z(x,z,t) = \int_{-\infty}^{\infty}\Big\{-E_1\xi_1 e^{-\xi_1 z} + E_2 ike^{-\xi_2 z} + E_3 ike^{-\xi_3 z} + E_4\xi_1 e^{\xi_1 z} + E_5 ike^{\xi_2 z} + E_6 ike^{\xi_3 z}\Big\}e^{i(kx+ft)}\hat{u}_0(f)\,df$$

- Non-dimensional parameters

$$A = X_0\sqrt{\frac{\mu}{B(\gamma+\varepsilon)}},\quad B = \frac{\alpha+\mu}{\alpha},\quad C = \frac{\gamma-\varepsilon}{\gamma+\varepsilon},\quad F = \frac{B-1}{A^2B}$$
$$C_1^2 = \frac{\lambda+2\mu}{\rho X_0^2 f_0^2},\quad C_2^2 = \frac{\mu}{\rho X_0^2 f_0^2},\quad C_3^2 = \frac{B}{B-1}C_2^2,\quad C_4^2 = \frac{\gamma+\varepsilon}{jX_0^2 f_0^2},\quad C_5^2 = \frac{\beta+2\gamma}{jX_0^2 f_0^2}$$

- Exponential factors

$$\nu_1 = \sqrt{k^2 - \frac{f^2}{C_1^2}},\quad \xi_1 = \sqrt{k^2 - \frac{f^2}{C_5^2} + \frac{4C_4^2}{FC_5^2}}$$
$$\nu_2 = \xi_2 = \sqrt{k^2 - A_m},\quad \nu_3 = \xi_3 = \sqrt{k^2 - A_p}$$
$$A_{p,m} = \frac{C_2^2+C_3^2}{2C_3^2C_4^2}f^2 - 2A^2 \pm \sqrt{\frac{(C_3^2-C_4^2)^2}{4C_3^4C_4^4}f^4 - \frac{2A^2(C_3^2C_2^2 - 2C_3^2C_4^2 + C_4^2C_2^2)}{C_2^2C_3^2C_4^2}f^2 + 4A^4}$$

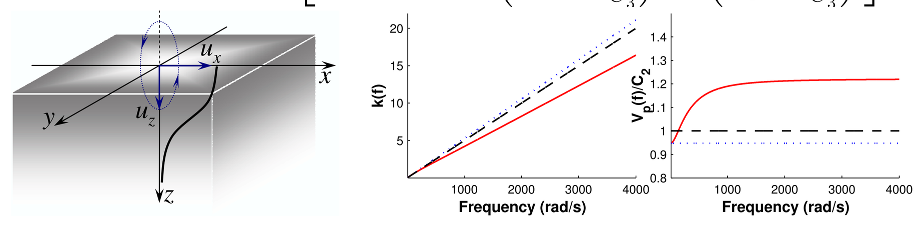
3 WAVES IN AN ELASTIC HALF-SPACE

- Amplitude of displacement components fades along the z-axis: $D_4 = D_5 = D_6 = 0$ and $E_4 = E_5 = E_6 = 0$
- The boundary conditions at the surface $z = 0$ require that normal forces and moments be zero and in the dimensionless form are

$$\sigma_{zx}|_{z=0} = 0,\quad \sigma_{zy}|_{z=0} = 0,\quad \sigma_{zz}|_{z=0} = 0$$
$$\mu_{zx}|_{z=0} = 0,\quad \mu_{zy}|_{z=0} = 0,\quad \mu_{zz}|_{z=0} = 0$$

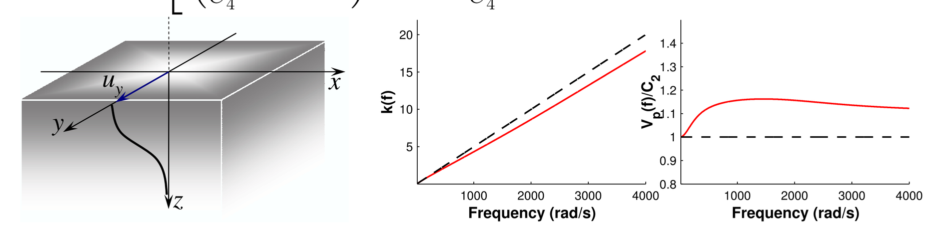
- Rayleigh wave with components u_x, u_z and ω_y is determined by the dispersion equation $\det(\mathcal{M}_r(\nu_1,\nu_2,\nu_3)) = 0$, where *(M. Kulesh et al., Journal of Applied Mechanics and Technical Physics, 46(2005), pp. 556-563)*.

$$\mathcal{M}_r(p_1,p_2,p_3) = \begin{bmatrix} 2k^2 - \frac{f^2}{C_2^2} & -2ikp_2 & -2ikp_3 \\ 2ikp_1 & 2k^2 - \frac{f^2}{C_2^2} & 2k^2 - \frac{f^2}{C_2^2} \\ 0 & p_2\left(A_m - \frac{f^2}{C_3^2}\right) & p_3\left(A_p - \frac{f^2}{C_3^2}\right) \end{bmatrix}$$



- Surface transverse wave with components u_y, ω_x and ω_z has the following dispersion equation: $\det(\mathcal{M}_t(\xi_1,\xi_2,\xi_3)) = 0$, where *(M. Kulesh et al., Acoustical Physics, 52(2006), pp. 186-193)*.

$$\mathcal{M}_t(p_1,p_2,p_3) = \begin{bmatrix} \frac{2ik}{1-B} & p_2\left(2 + \frac{A_mC_4^2 - f^2}{2A^2C_4^2}\right) & p_3\left(2 + \frac{A_pC_4^2 - f^2}{2A^2C_4^2}\right) \\ ikp_1(1+C) & p_2^2 + k^2C & p_3^2 + k^2C \\ \left(\frac{C_2^2}{C_4^2} - C - 1\right)k^2 - p_1^2\frac{C_2^2}{C_4^2} & ikp_2(1+C) & ikp_3(1+C) \end{bmatrix}$$



Note that in the half-space whose dynamic behavior is described by the Cosserat model, in addition to the elliptic surface Rayleigh wave, it is also possible to observe another wave type, a surface wave whose one component is parallel to the boundary surface and perpendicular to the propagation direction as revealed by our soltuions to the equations of motion. Both Rayleigh and surface transverse wave have the dispersive character of propagation in the half-space, observation that cannot be explained by the classical elasticity theory.

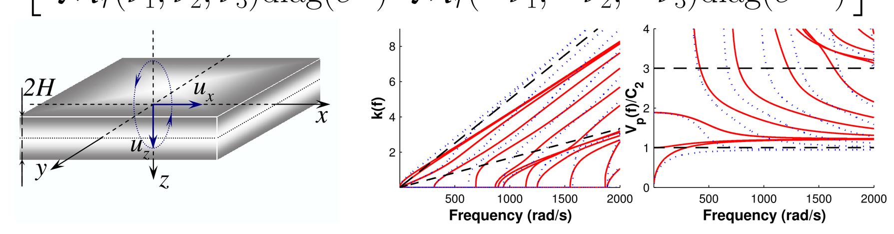
4 WAVES IN A THIN LAYER

- Let us consider a free loaded plate with the thickness $2H$, characteristic length $X_0 = H$
- The boundary conditions at the surfaces $z = \pm 1$ require that normal forces and moments are zero and in the dimensionless form are

$$\sigma_{zx}|_{z=\pm1} = 0,\quad \sigma_{zy}|_{z=\pm1} = 0,\quad \sigma_{zz}|_{z=\pm1} = 0$$
$$\mu_{zx}|_{z=\pm1} = 0,\quad \mu_{zy}|_{z=\pm1} = 0,\quad \mu_{zz}|_{z=\pm1} = 0$$

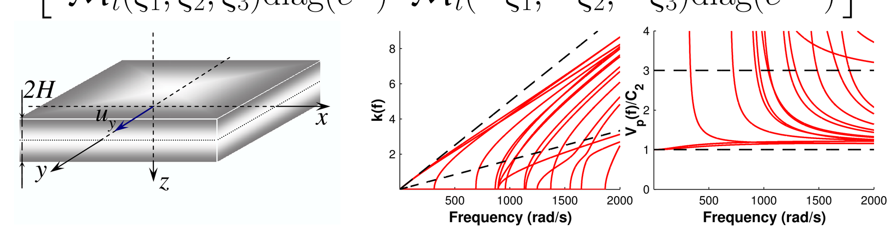
- Lamb wave with components u_x, u_z and ω_y is described by the dispersion equation

$$\det\begin{bmatrix} \mathcal{M}_r(\nu_1,\nu_2,\nu_3)\operatorname{diag}(e^{-\nu_n}) & \mathcal{M}_r(-\nu_1,-\nu_2,-\nu_3)\operatorname{diag}(e^{\nu_n}) \\ \mathcal{M}_r(\nu_1,\nu_2,\nu_3)\operatorname{diag}(e^{\nu_n}) & \mathcal{M}_r(-\nu_1,-\nu_2,-\nu_3)\operatorname{diag}(e^{-\nu_n}) \end{bmatrix} = 0$$



- Transverse wave with components u_y, ω_x and ω_z has the following dispersion equation:

$$\det\begin{bmatrix} \mathcal{M}_t(\xi_1,\xi_2,\xi_3)\operatorname{diag}(e^{-\xi_n}) & \mathcal{M}_t(-\xi_1,-\xi_2,-\xi_3)\operatorname{diag}(e^{\xi_n}) \\ \mathcal{M}_t(\xi_1,\xi_2,\xi_3)\operatorname{diag}(e^{\xi_n}) & \mathcal{M}_t(-\xi_1,-\xi_2,-\xi_3)\operatorname{diag}(e^{-\xi_n}) \end{bmatrix} = 0$$



- In the expressions above $\operatorname{diag}(e^{-p_k})$ is the diagonal matrix.

Note that a qualitatively new wave mode with only one displacement component exists in a free loaded plate within the framework of the Cosserat medium besides well investigated Lamb wave. As in the case of surface transverse wave this new mode also does not have any analogy in the classical elasticity theory.

5 MODELING OF SURFACE WAVE PROPAGATION USING CONTINUOUS WAVELET TRANSFORMS

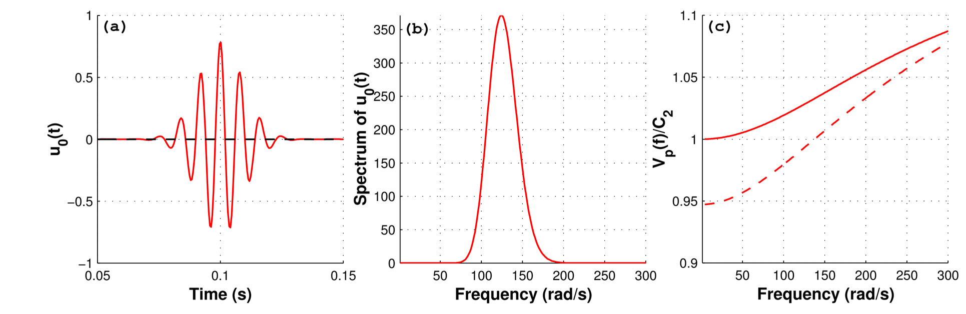
Let us use the asymptotic propagator in the wavelet space *(M. Kulesh et al., Pure Appl. Geophys, 162(2005), pp. 843-855)*. This propagator is a mathematical model to establish a link between the continuous wavelet transform of a signal and its propagated counterpart in a dispersive medium:

$$\mathcal{W}_g u_2(t,f) = \left|\mathcal{W}_g u_1\left(t - \tfrac{D}{V_g(f)}, f\right)\right| \exp\left[i\arg\mathcal{W}_g u_1\left(t - \tfrac{D}{V_p(f)}, f\right)\right],$$

where

- $\mathcal{W}_g u_1(t,f)$ is the wavelet transform of the displacement component $u_n(x_1,0,t)$ at the surface and at the distance x_1 from the source
- $\mathcal{W}_g u_2(t,f)$ is the wavelet spectrum of propagated signal at the distance $x_2 = x_1 + D$
- Signals $u_1(t)$ and $u_2(t)$ are real in case of the transverse wave, but for the Rayleigh wave these signals can be complex: $u_1(t) = u_x(t) + iu_z(t)$ *(M. Kulesh et al., Acoustical Physics, 51(2005), pp. 425-434)*.
- The analyzing wavelet has the linear phase (with time-derivative approximately equal to 2π, as it is the case for the Morlet wavelet)
- $V_p(f)$ and $V_g(f)$ are the phase and group velocities

Figure 2: Complex Cauchy wavelet as source signal and dispersive parameters of medium

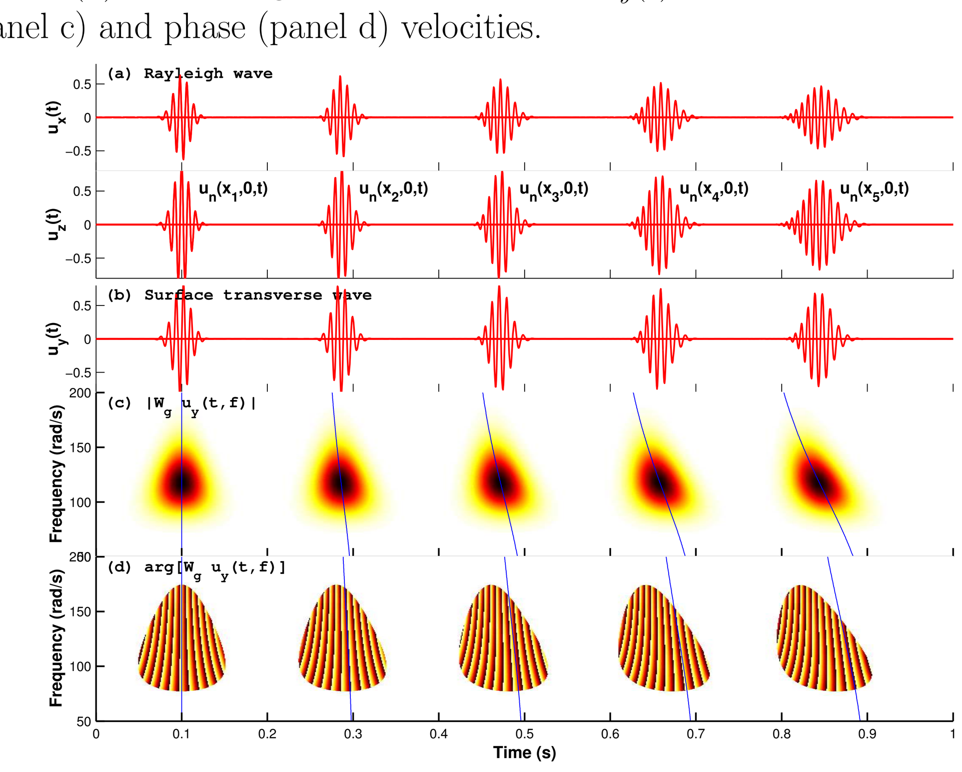


We can use the wavelet transform in the following form:

$$\mathcal{W}_g S(t,a) = \int_{-\infty}^{+\infty}\frac{1}{a}g^*\Big(\frac{\tau-t}{a}\Big)S(\tau)\,d\tau,$$
$$S(t) = \frac{1}{C_{g,m}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{a^2}m\left(\frac{t-\tau}{a}\right)\mathcal{W}_g S(\tau,a)\,d\tau\,da,$$
$$C_{g,m} = \int_0^{+\infty}\left(\hat{g}^*(f)\hat{m}(f) + \hat{g}^*(-f)\hat{m}(-f)\right)\frac{df}{f},$$

where $g(\tau)$ is a mother wavelet, $m(t)$ is a synthesizing wavelet, $a = f_0/f \in \mathbb{R}$ is the scale parameter (dilation), $t \in \mathbb{R}$ is the location parameter (translations) and $(\cdot)^*$ indicates the complex conjugation.

Figure 3: (a,b) Synthetic three-component seismogram, (c) wavelet modulus and (d) phase images for the component $u_y(t)$. Solid lines are the group (panel c) and phase (panel d) velocities.



Note that the group velocity is a function that "deforms" the image of the absolute value of the source signal's wavelet spectrum, the phase velocity "deforms" the image of the wavelet spectrum's phase. Thus, the full dispersion characteristics are explicitly expressed and therefore can be easily extracted from the modulus and phase of wavelet transform.

CONCLUSION

- In this study we have obtained four solutions corresponding to Rayleigh wave and surface transverse wave in a half-space as well as Lamb wave and transverse wave in a thin layer.
- These solutions can be divided into two groups, one of which corresponds to the well-investigated elliptical wave and the other — to the transverse wave with depth-dependent decay which does not have any analogy in the classical theory of elasticity.
- We have compared the solution for all observed waves to classical case with help of numerical illustrations.
- We have also considered a model that binds the wavelet spectra of two signals at different distance. This model allows us to determine the phase and group velocities of Rayleigh and surface transverse waves.

WORK IN PROGRESS

- Analysis of displacemet components for the Lamb and transverse waves in a thin layer.
- Construction of analytical solution (dispersion equation and displacement components) for the Stoneley wave.
- Retrieval the publications with experimental results for the wave propagation problem in medium with microstructure.