# Rayleigh-type waves for the isotropic elastic reduced Cosserat half-space. Application in rocks and soils

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## ABSTRACT

We continue in this paper the work on the modelisation of rocks and soils in terms of the elastic reduced Cosserat continuum. The basic idea of the model is to take into account the mictrostructure of rocks and soils which influences the wave propagation, for some frequency range trapping the energy of the propagating wave by their proper rotational motion. We consider the Rayleigh wave for the isotropic case, using analytical and numerical methods. Instead of a straight line in the classical medium, we obtain two dispersion curves. The polarization differs both from the case of classical medium and the case of Cosserat continuum with couple stresses. We observe for some frequency range a strong frequency dependence. There is a forbidden band of frequencies, lying below the analogous forbidden band for an unbounded medium. It indicates the possibility of localization phenomena. For the upper branch of dispersion relation, there is also a forbidden domain of wave numbers: long waves may propagate only with one frequency. Far from the domain of frequencies where the microstructure influences the wave propagation, the medium behaves analogously to the classical one.

## 1 ROCKS AND COMPRESSED SOILS AS REDUCED COSSERAT CONTINUUM — MODEL

We consider a heterogeneous elastic medium with inclusions as a homogeneous reduced Cosserat continuum, whose point-bodies may rotate and move.

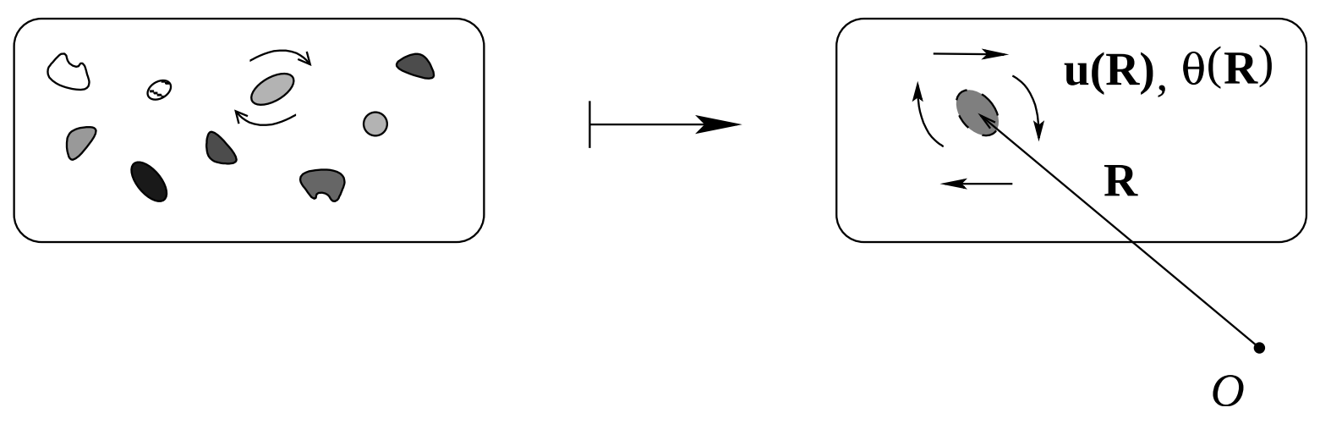


Figure 1. Rock as an enriched continuum

Rotations and translations are independent. The medium reacts on the rotation of a point-body relatively to the background continuum, but there is no “rotational spring” trying to reduce the relative turn of point-bodies $\Longrightarrow$ stress tensor is asymmetric, but couple stress is zero.

## 2 REDUCED COSSERAT CONTINUUM — EQUATIONS OF MOTION

- $\boldsymbol{u} = \{u_x, u_y, u_z\}$ and $\boldsymbol{\theta} = \{\theta_x, \theta_y, \theta_z\}$ are independent displacement and rotation vectors.
- Constitutive equation:

$$\boldsymbol{\tau} = \lambda \boldsymbol{E} \nabla \cdot \boldsymbol{u} + 2\mu (\nabla \boldsymbol{u})^S + 2\alpha(\nabla \boldsymbol{u} + \boldsymbol{\theta} \times \boldsymbol{E})^A,$$

- Balance of force and torque:

$$\nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{K} = \rho \ddot{\boldsymbol{u}}, \qquad \boldsymbol{\tau}_\times + \rho \boldsymbol{L} = \boldsymbol{I} \cdot \dot{\boldsymbol{\theta}}.$$

- Consider $\boldsymbol{I} = I\boldsymbol{E}$, no external loads: $\boldsymbol{K} = \boldsymbol{0}$, $\boldsymbol{L} = \boldsymbol{0}$. We have the equations of motion in translational and angular displacements

$$\boxed{\begin{gathered}(\lambda + 2\mu)\nabla\nabla \cdot \boldsymbol{u} - (\mu + \alpha)\nabla \times (\nabla \times \boldsymbol{u}) + 2\alpha\, \nabla \times \boldsymbol{\theta} = \rho \ddot{\boldsymbol{u}},\\ 2\alpha\, \nabla \times \boldsymbol{u} - 4\alpha\, \boldsymbol{\theta} = I\ddot{\boldsymbol{\theta}}.\end{gathered}}$$

- In the equations of motion $\mu$ and $\lambda$ are the Lame constants, $\alpha$ is the physical constants of a material in the framework of the reduced Cosserat medium, $\rho$ is the density, $I$ is the moment of inertia density, $(.)^{(S)}$ and $(.)^{(A)}$ denote the symmetric and antisymmetric parts of tensor and $\boldsymbol{E}$ is the identity tensor.

## 3 PROBLEM STATEMENT FOR THE RAYLEIGH WAVE

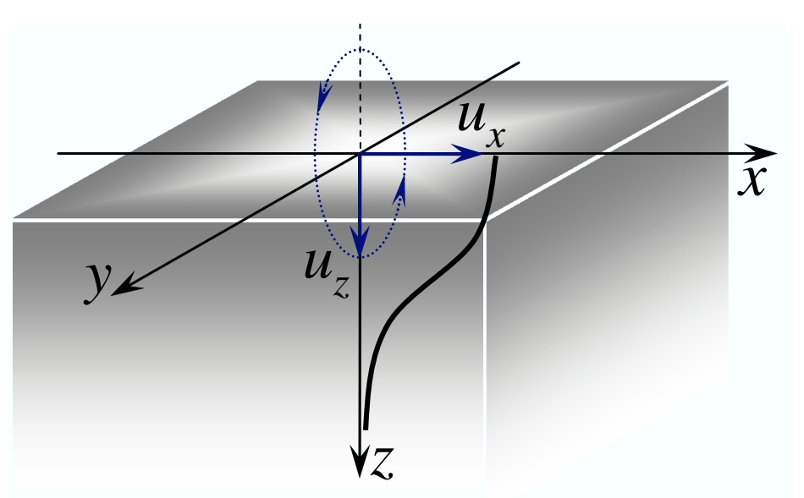


Figure 2. Rayleigh wave in reduced Cosserat continuum

- Consider surface plane wave in the free elastic half-space.
  $z$ (axis $\boldsymbol{i}_3$) — depth, $x, y$ (axes $\boldsymbol{i}_1, \boldsymbol{i}_2$) — plane co-ordinates.
- Boundary conditions at the surface are

$$\boldsymbol{\tau}_{zx}|_{z=0} = 0, \quad \boldsymbol{\tau}_{zy}|_{z=0} = 0, \quad \boldsymbol{\tau}_{zz}|_{z=0} = 0.$$

- We look for the solution of equations of motion in the form of

$$\boldsymbol{u} = \boldsymbol{U}(z)e^{i(kx-\omega t)}, \quad \boldsymbol{\theta} = \boldsymbol{\Theta}(z)e^{i(kx-\omega t)},$$

  where $\boldsymbol{U}(z)$ and $\boldsymbol{\Theta}(z)$ are decaying functions depending on frequency $\omega$ and wavenumber $k$. Only real parts of these complex-valued functions have the physical meaning.

## 4 DISPERSION RELATION FOR THE RAYLEIGH WAVE

We obtain two independent systems from the equations of motion: the system for $U_x(z), U_z(z), \Theta_y(z)$ and the system for $U_y(z), \Theta_x(z), \Theta_z(z)$. Rayleigh-type: solutions of the 1st system, decreasing with depth $z$.

$$U_x(z) = D_1 ike^{-\nu_1 z} + D_2\nu_2 e^{-\nu_2 z}, \quad U_z(z) = -D_1\nu_1 e^{-\nu_1 z} + D_2 ike^{-\nu_2 z},$$
$$\Theta_y(z) = D_2 \frac{\omega^2}{2C_s^2(1-\omega^2/\omega_1^2)} e^{-\nu_2 z},$$

where

$$\boxed{\nu_1 = \sqrt{k^2 - \frac{\omega^2}{C_p^2}}, \quad \nu_2 = \sqrt{k^2 - \frac{\omega^2}{C_s^2}\left(\frac{1-\omega^2/\omega_0^2}{1-\omega^2/\omega_1^2}\right)},}$$

$$\omega_0^2 = \frac{4\alpha}{I}, \quad \omega_1^2 = \frac{\omega_0^2}{1+\alpha/\mu}, \quad C_p^2 = \frac{\lambda + 2\mu}{\rho}, \quad C_s^2 = \frac{\mu}{\rho}.$$

Boundary condition on the free surface: $\boldsymbol{i}_3 \cdot \boldsymbol{\tau} = \boldsymbol{0} \Longrightarrow$ dispersion relation for the Rayleigh wave (the modified Rayleigh equation):

$$\boxed{4k^2\nu_1\nu_2 = (2k^2 - \omega^2/C_s^2)^2.}$$

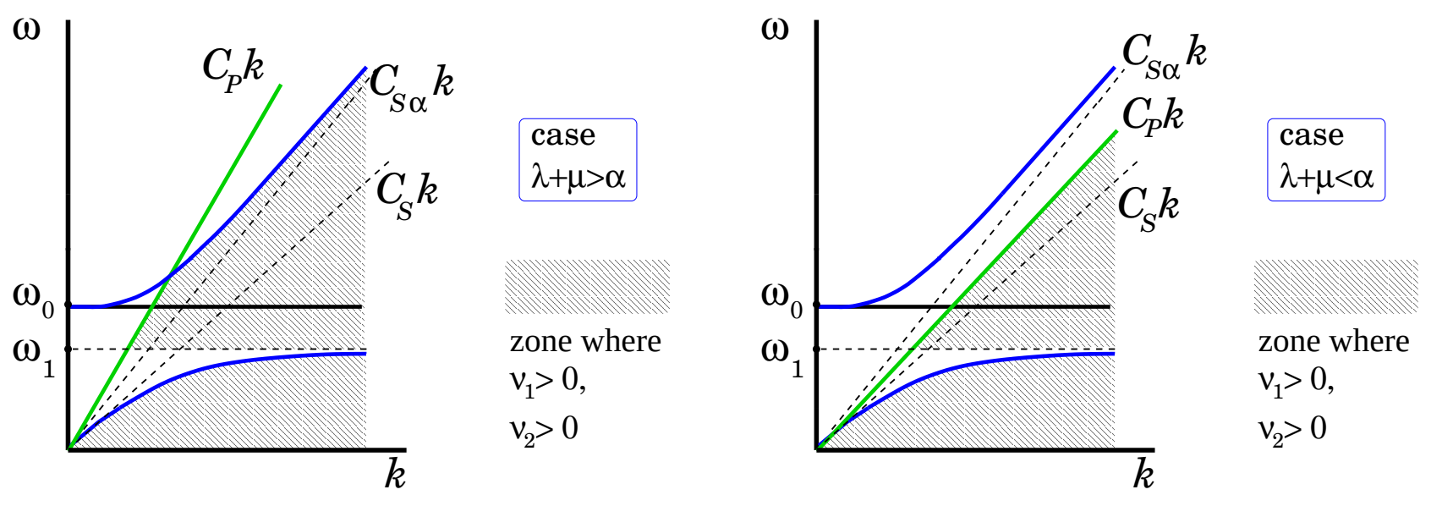


Figure 3. Zones of existence for Rayleigh waves: $\nu_1, \nu_2 \in \Re$. Here $C_{S\alpha}^2 = (\mu + \alpha)/\rho$.

Equations $\nu_1 = 0$, $\nu_2 = 0$ are the dispersions relations for the 3D plane wave, as in the classical case.

In the classical case the Rayleigh wave exists for all $\omega$.

## 5 ANALYTICAL SOLUTION

Another form of the Rayleigh equation ($k = \omega\sqrt{x/2}/C_s$):

$$a(\omega)x^3 + b(\omega)x^2 + 4x - 1 = 0,$$

where

$$a(\omega) = 2\left(1 - \frac{C_s^2}{C_p^2} + \frac{\omega^2}{(1+\mu/\alpha)(\omega^2-\omega_1^2)}\right),$$
$$b(\omega) = 2\left(2\frac{C_s^2}{C_p^2} - 3 - \frac{2C_s^2\omega^2}{C_p^2(1+\mu/\alpha)(\omega^2-\omega_1^2)}\right),$$

$$c(\omega) = \sqrt[3]{27a(\omega)^2 - 2b(\omega)^3 + 36a(\omega)b(\omega) + d(\omega)},$$

$$d(\omega) = 3\sqrt{3}a(\omega)\sqrt{27a(\omega)^2 - 4b(\omega)^3 - 16b(\omega)^2 + 72a(\omega)b(\omega) + 256a(\omega)}.$$

Solution of the Rayleigh equation:

$$\boxed{k(\omega) = \frac{\omega}{C_s}\sqrt{\frac{\sqrt[3]{4}c(\omega)}{12a(\omega)} - \frac{b(\omega)}{6a(\omega)} + \frac{\sqrt[3]{2}(b(\omega)^2 - 12a(\omega))}{6a(\omega)c(\omega)}}.}$$

## 6 ANALYSIS OF THE SOLUTION

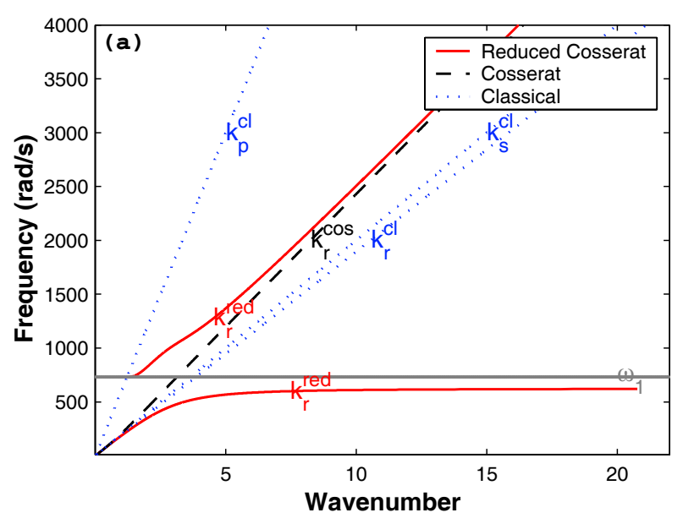


Figure 4. Dispersion graph for the Rayleigh wave

Two cases investigated:

- $k \to \infty, \quad \omega = O(1)$
- $k \to \infty, \quad \omega \to \infty$

1) case $k \to \infty$, $\omega = O(1)$. From the solution of the Rayleigh equation we have

$$\omega \to \omega_2 = \omega_1\sqrt{\frac{A}{1+A}}, \quad A = \left(1 + \frac{\mu}{\alpha}\right)\left(1 - \frac{\mu}{\lambda + 2\mu}\right) > \frac{1}{2}.$$

The boundary frequency $\omega_2 \in (\omega_1/\sqrt{3}; \omega_1)$ for any elastic constants $\Longrightarrow$ at $k \to \infty$ it lies in the “allowed zone” $\nu_1, \nu_2 \in \Re$. This is a horizontal asymptote — the lower boundary of the forbidden band.

2) case $k \to \infty$, $\omega \to \infty$. Looking for the solution $\omega \approx C_\infty k$ in this limit, we obtain the following cubic equation for $\xi = C_\infty^2/C_s^2$:

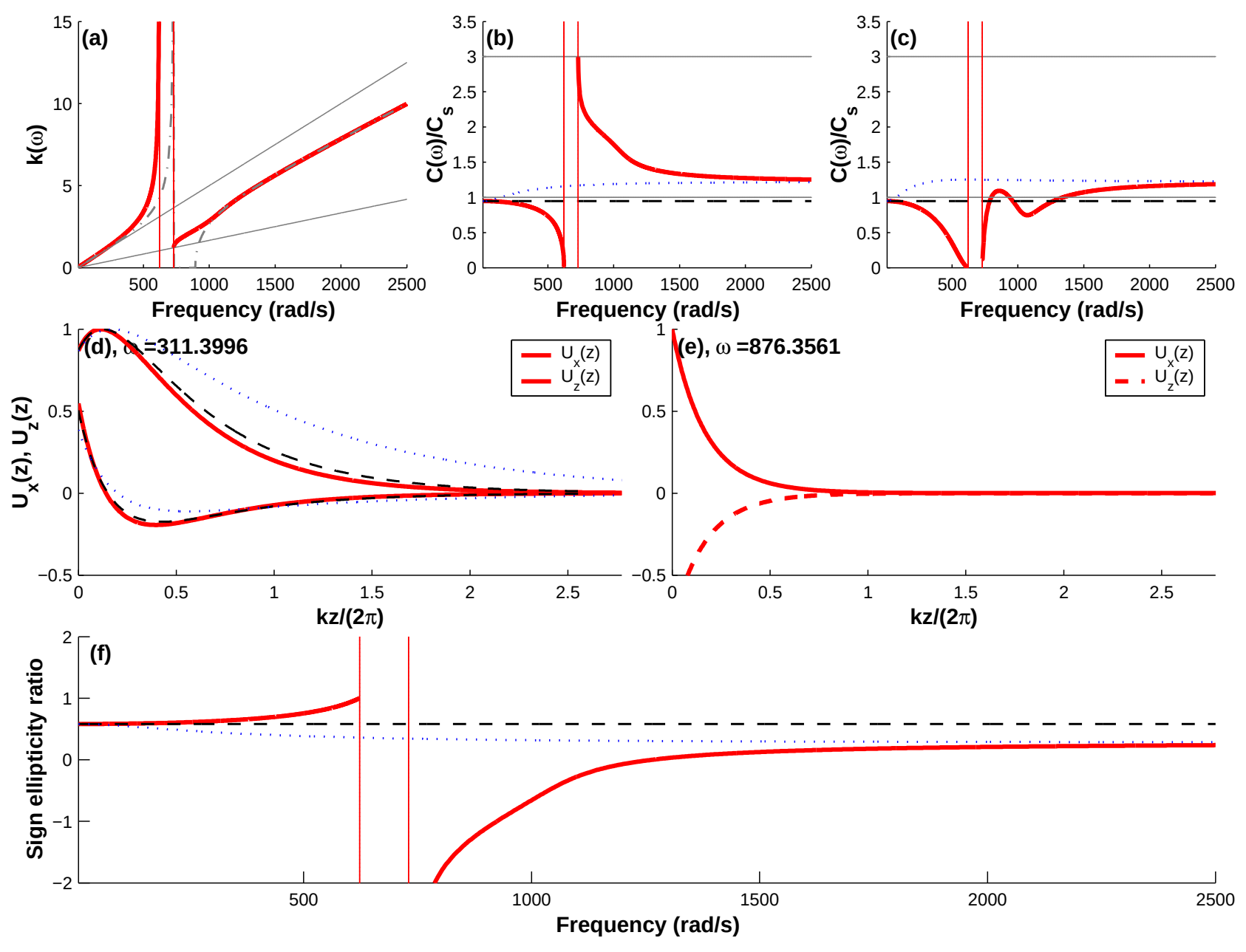
$$g(\xi) = \xi^3 - 8\xi^2 + 8\xi(3 - 2pq) - 16(2 - (p + q)) = 0.$$

where $p = C_s^2/C_p^2$, $q = \mu/(\mu + \alpha) \in (0; 1)$. If $\alpha = 0$, this equation coincides with the classical case. This equation has only one real (and positive) root. One can prove $C_\infty < C_{s\alpha}$, $C_\infty < C_p$ (the asymptote lies in the allowed zone $\nu_1, \nu_2 \in \Re$).

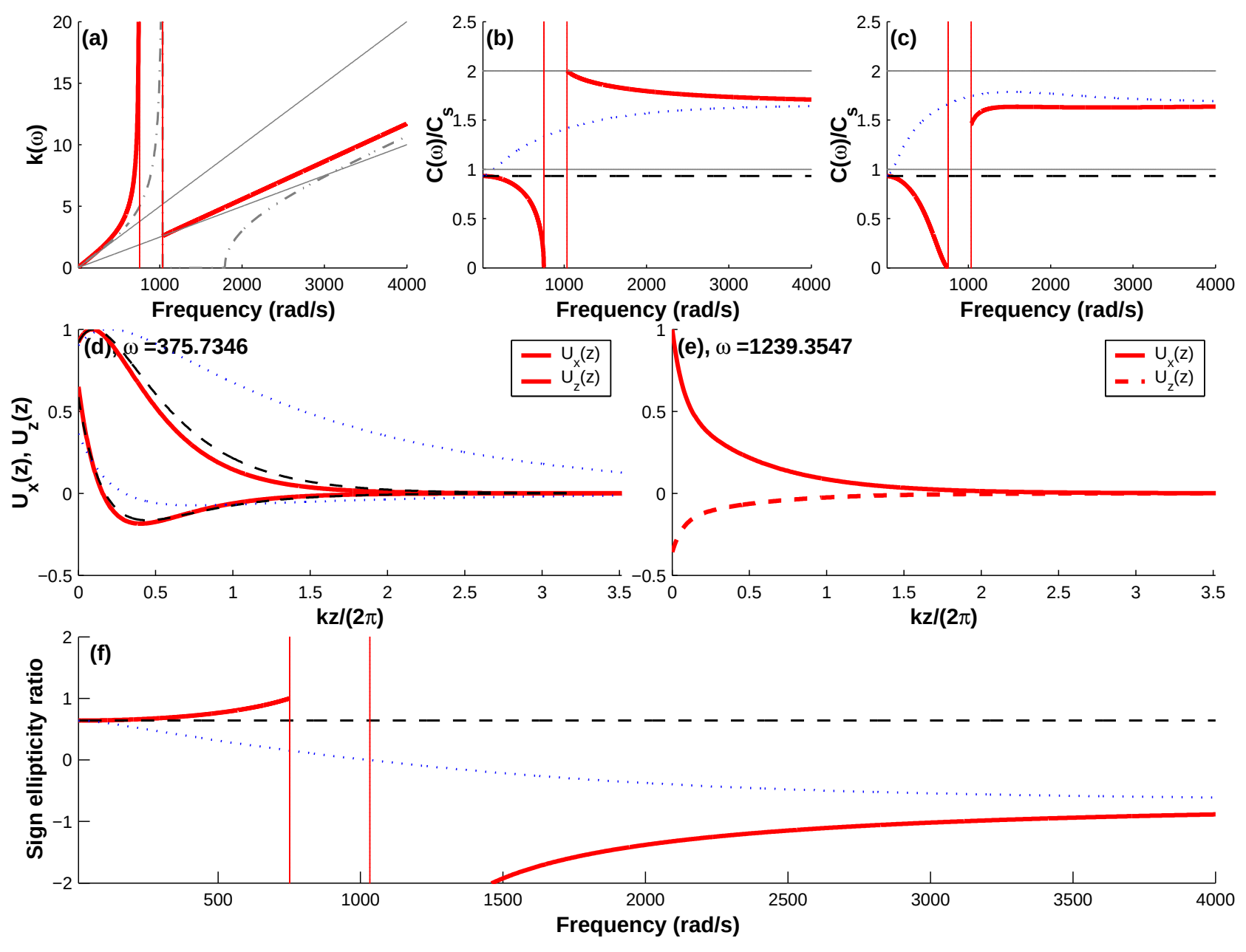
$$C_\infty < C_s \Longleftrightarrow \frac{\alpha}{\mu} < \frac{2 + \lambda/\mu}{14 + 15\lambda/\mu} \in (1/15; 1/7).$$

## 7 NUMERICAL SIMULATIONS

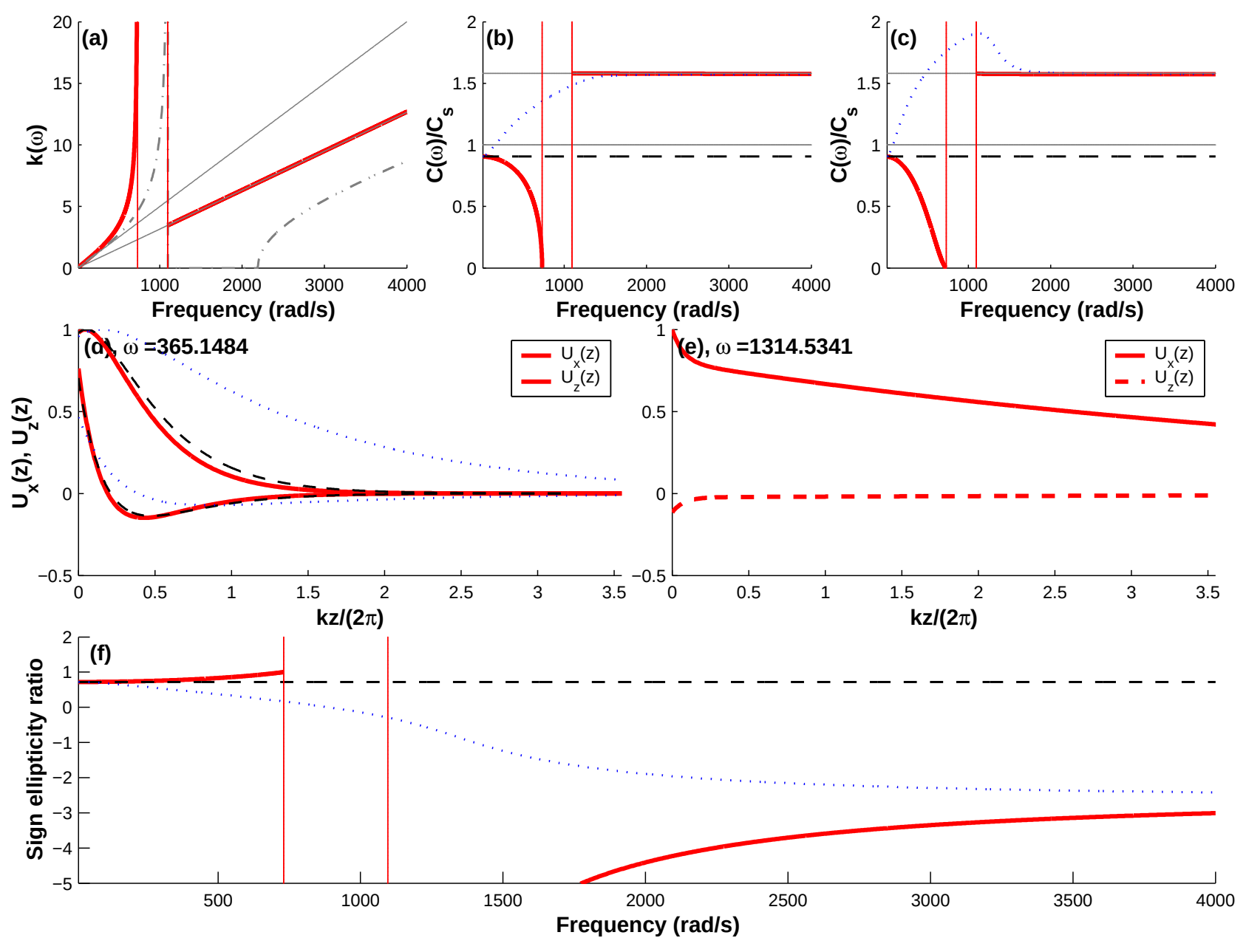
- The first set of matherial constants: $\lambda = 2.8 \cdot 10^{10}$, $\mu = 4.0 \cdot 10^9$, $\rho = 1.0 \cdot 10^5$, $\alpha = 2.0 \cdot 10^9$, $j = 10 \cdot 10^3$, $C_s = 200$, $C_p = 600$, $\omega_1 = 730.2967$, $\omega_2 = 622.7992$.



- The second set of matherial constants: $\lambda = 8.0 \cdot 10^9$, $\mu = 4.0 \cdot 10^9$, $\rho = 1.0 \cdot 10^5$, $\alpha = 8.0 \cdot 10^9$, $j = 10 \cdot 10^3$, $C_s = 200$, $C_p = 400$, $\omega_1 = 1.0328 \cdot 10^3$, $\omega_2 = 751.4691$.



- The third set of matherial constants: $\lambda = 2.0 \cdot 10^9$, $\mu = 4.0 \cdot 10^9$, $\rho = 1.0 \cdot 10^5$, $\alpha = 1.2 \cdot 10^{10}$, $j = 10 \cdot 10^3$, $C_s = 200$, $C_p = 316.2278$, $\omega_1 = 1.0954 \cdot 10^3$, $\omega_2 = 730.2967$.



In these pictures:

- Red bold lines describe the solution for reduced Cosserat medium,
- Red vertical lines correspond to critical frequencies $\omega_1$ and $\omega_2$,
- Grey horizontal lines correspond to velocities $C_s$ and $C_p$,
- Hatch black lines depict the solution for Rayleight wave in the classical case,
- Dotted blue lines depict the solution within the wramework of Cosserat continumm *(M. Kulesh et al. Acoustical Physics, 2006, Vol. 52, No. 2, pp. 186-193.)*

## CONCLUSION

- There is a forbidden zone $(\omega_2; \omega_1)$ where the Rayleigh wave does not propagate (similar to the 3D case; unlike the classical and full Cosserat medium cases). The energy of wave is caught by rotation and localised.
- We observe a strong dispersion in some frequency domains (similar to the 3D case and to the complete Cosserat medium case, unlike the classical case).
- There is a domain of “forbidden wave numbers” for one of the branches of the dispersion relation.
- At frequency $\omega = \omega_1$ we may observe independent rotational vibrations of point-bodies.
- There is no surface transversal wave decreasing with depth (similar to the classical case, unlike the full Cosserat case).

## WORK IN PROGRESS

- Completion of the theoretical proof of the existence of forbidden zone
- Analysis of the neighbourhood of the point $(\omega_1/C_p; \omega_1)$
- Retrieval the publications with experimental results for the wave propagation problem in medium with microstructure

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