

Analytical and numerical investigation of dynamic problems in the framework of asymmetric elasticity theory

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ABSTRACT

In 2009 the world scientific community will mark the 100th anniversary of publication of the fundamental Cosserat brothers' study, in which they defined theoretical grounds of the asymmetric (couple-stress) elasticity theory for isotropic solid. However, up to the present we have no a clear understanding as to the role and the place of this theory in the continuum mechanics. One way to clarify this question is to develop a thoroughly elaborated experiment which could provide evidence for couple-stress behavior of elastic materials and identify all material constants entering the constitutive relations of the Cosserat continuum model. In this paper, the wave problems and the problem of natural vibrations are considered in search of such experimental scheme.

This paper is concerned with construction of general motion equation for a linear isotropic Cosserat medium. This equation yields a general analytical solution to the linear wave problem in terms of displacements, from which one can derive solutions for Rayleigh, Lamb and Stoneley waves and also for a plane volume wave in the Cosserat space. A number of marked differences of the obtained solutions from the classical symmetric case have been demonstrated. The finite-element algorithm has been developed to solve problems on natural vibrations of elastic bodies in the framework of the developed model. This algorithm has been applied to some practical problems on natural vibrations. The obtained results can be used to identify an important characteristic of the considered system - the spectrum of natural frequencies, which is an experimentally measurable macroparameter responding to the couple-stress properties of the medium.

The obtained analytical and numerical solutions are analyzed for their applicability to experiments on identification of elastic constants and recording the fact of couple stress behavior of elastic materials.

1 ASYMMETRIC THEORY OF ELASTICITY

Basic relations of the elastic Cosserat medium (*W. Nowacki. Teoria niesymetrycznej sprzyzystosci. Warszawa, PNW (1971)*):

Force and moment of force $\mathbf{p} = \mathbf{n} \cdot \boldsymbol{\sigma}$, $\mathbf{m} = \mathbf{n} \cdot \tilde{\boldsymbol{\mu}}$

Displacement vector $\mathbf{u} = \{u_x, u_y, u_z\}$

Rotation vector $\boldsymbol{\omega} = \{\omega_x, \omega_y, \omega_z\}$

Asymmetric strain tensor $\tilde{\boldsymbol{\gamma}} = \nabla \mathbf{u} - \tilde{\mathbf{E}} \cdot \boldsymbol{\omega}$

Asymmetric torsion bending tensor $\tilde{\boldsymbol{\chi}} = \nabla \boldsymbol{\omega}$

Asymmetric stress tensor $\boldsymbol{\sigma} = 2\mu\tilde{\boldsymbol{\gamma}}^{(S)} + 2\alpha\tilde{\boldsymbol{\gamma}}^{(A)} + \lambda I_1(\tilde{\boldsymbol{\gamma}})\tilde{\mathbf{e}}$

Asymmetric couple-stress tensor $\tilde{\boldsymbol{\mu}} = 2\gamma\tilde{\boldsymbol{\chi}}^{(S)} + 2\varepsilon\tilde{\boldsymbol{\chi}}^{(A)} + \beta I_1(\tilde{\boldsymbol{\chi}})\tilde{\mathbf{e}}$

Equilibrium equations

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{X} = \rho \ddot{\mathbf{u}}, \quad \tilde{\boldsymbol{\sigma}}^T : \tilde{\mathbf{E}} + \nabla \cdot \tilde{\boldsymbol{\mu}} + \mathbf{Y} = j \ddot{\boldsymbol{\omega}}$$

Equations of motion

$$(\lambda + 2\mu)\text{grad div } \mathbf{u} - (\mu + \alpha)\text{rot rot } \mathbf{u} + 2\alpha \text{rot } \boldsymbol{\omega} + \mathbf{X} = \rho \ddot{\mathbf{u}}, \\ (\beta + 2\gamma)\text{grad div } \boldsymbol{\omega} - (\gamma + \varepsilon)\text{rot rot } \boldsymbol{\omega} + 2\alpha \text{rot } \mathbf{u} - 4\alpha \boldsymbol{\omega} + \mathbf{Y} = j \ddot{\boldsymbol{\omega}},$$

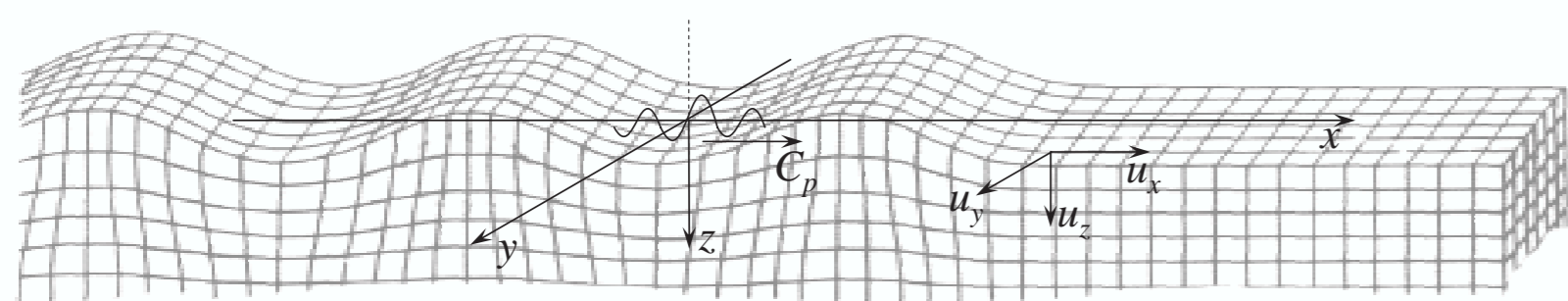
Variational Equation

$$\int_V (\boldsymbol{\sigma} : \delta \tilde{\boldsymbol{\gamma}} + \tilde{\boldsymbol{\mu}} : \delta \tilde{\boldsymbol{\chi}}) dV - \int_V (\rho \ddot{\mathbf{u}} \cdot \delta \mathbf{u} + j \ddot{\boldsymbol{\omega}} \cdot \delta \boldsymbol{\omega}) dV = 0$$

where μ and λ are the Lamé constants, α, β, γ and ε are the physical constants of a material in the framework of the Cosserat medium, ρ is the density, j is the moment of inertia density, $(\cdot)^{(S)}$ and $(\cdot)^{(A)}$ denote the symmetric and antisymmetric parts of tensor and $\tilde{\mathbf{E}}$ is the Levi-Civita tensor of the third rank.

2 PLANE WAVES SOLUTION

Representation of the solution for displacement and rotation vectors components in the form of a wave packet:



$$u_n(x, z, t) = \int_{-\infty}^{\infty} U_n(z) e^{i(kx(f)+ft)} \hat{u}_0(f) df, \\ \omega_n(x, z, t) = \int_{-\infty}^{\infty} W_n(z) e^{i(kx(f)+ft)} \hat{u}_0(f) df,$$

where $k(f)$ is the wavenumber, f is the circular frequency, t is the time, $U_n(z)$ and $W_n(z)$ are amplitude functions depending on depth and frequency. $\hat{u}_0(f)$ is the complex spectral function corresponding to the Fourier spectrum of a source signal and determines the wavepacket form.

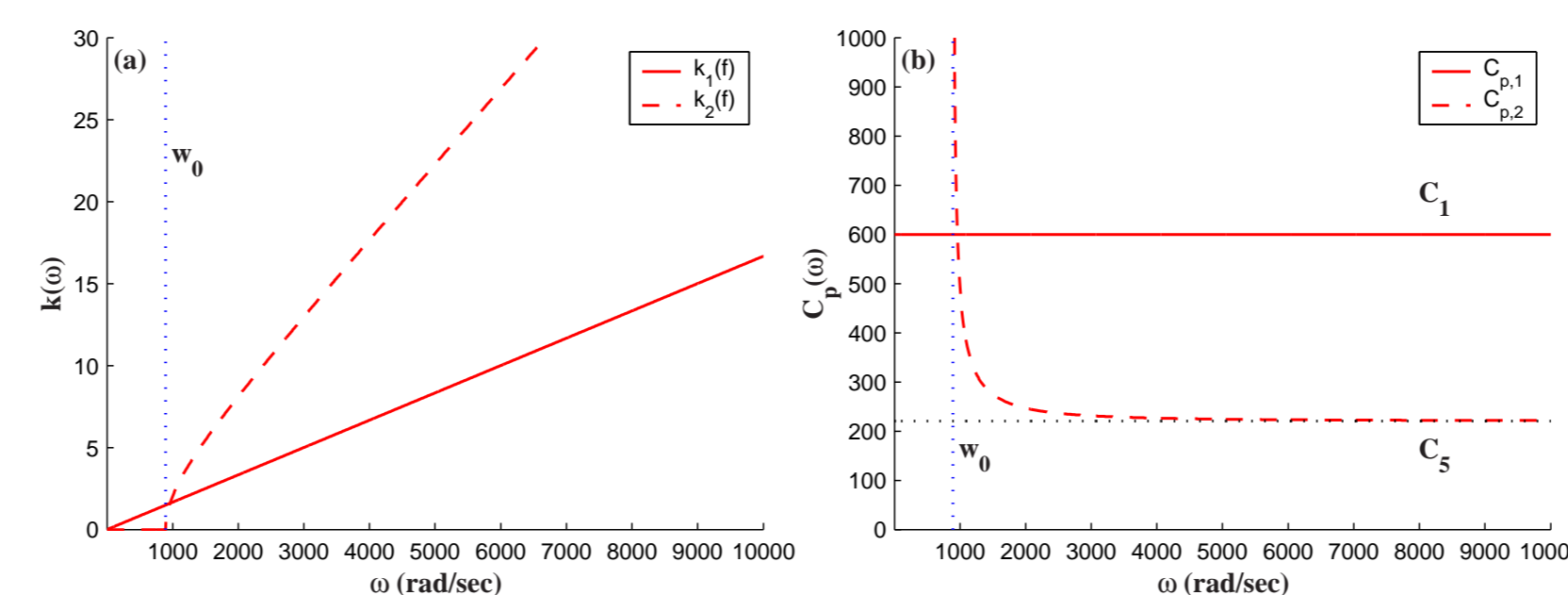
3 PLANE BODY WAVES

Amplitude of displacement components are constant along the z -axis.

Plane longitudinal wave has two modes, the rotation mode is dispersive and has an forbidden zone:

$$k_1(f) = \frac{f}{C_1}, \quad k_2(f) = \sqrt{\frac{f^2}{C_5^2} - k_0^2}, \\ C_1^2 = \frac{\lambda+2\mu}{\rho X_0^2 f_0^2}, \quad C_5^2 = \frac{\beta+2\gamma}{j X_0^2 f_0^2}, \quad u_0 = 2\sqrt{\frac{\alpha}{j}}, \quad k_0 = 2\sqrt{\frac{\alpha}{\beta+2\gamma}}.$$

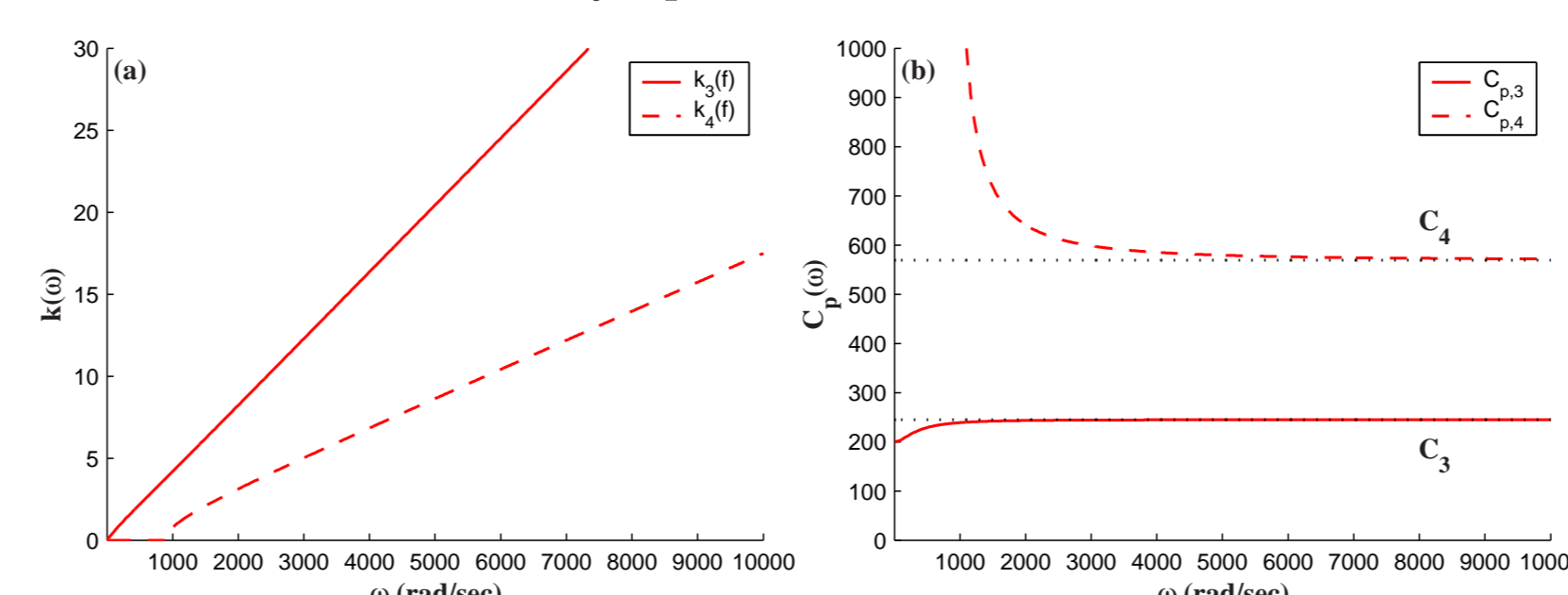
where X_0 - characteristic length, f_0 - characteristic frequency, C_1 is a phase velocity of displacement wave, and C_5 is a phase velocity of rotation wave. The rotation wave has no analogy in classical theory.



Plane transversal wave has two wave modes k_3 and k_4 , both are dispersive and one of them has a forbidden zone:

$$k_3(f) = \sqrt{A_p}, \quad k_4(f) = \sqrt{A_m}, \\ A^2 = X_0^2 \frac{\mu\alpha}{(\mu+\alpha)(\gamma+\varepsilon)}, \quad C_2^2 = \frac{\mu}{\rho X_0^2 f_0^2}, \quad C_3^2 = \frac{\mu+\alpha}{\rho X_0^2 f_0^2}, \quad C_4^2 = \frac{\gamma+\varepsilon}{j X_0^2 f_0^2}, \\ A_{p,m} = \frac{C_3^2 + C_4^2}{2C_3^2 C_4^2} f^2 - 2A^2 \pm \sqrt{\frac{(C_3^2 - C_4^2)^2}{4C_3^2 C_4^2} f^4 - 2A^4 \frac{C_3^2 C_4^2 - 2C_3^2 C_4^2 + C_3^2 C_4^2}{C_3^2 C_3^2 C_4^2} f^2 + 4A^4}$$

where C_3 and C_4 are the asymptotic velocities of transversal wave modes.



4 WAVES IN AN ELASTIC HALF-SPACE

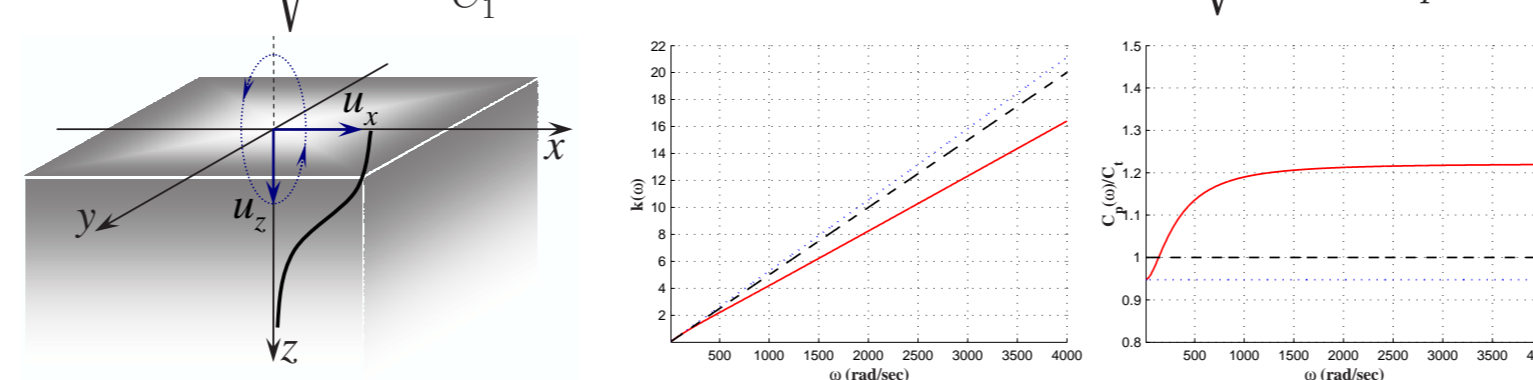
Amplitude of displacement components fades along the z -axis.

The boundary conditions at the surface $z = 0$ require that normal forces and moments be zero and in the dimensionless form are

$$\sigma_{zx}|_{z=0} = 0, \quad \sigma_{zy}|_{z=0} = 0, \quad \sigma_{zz}|_{z=0} = 0 \\ \mu_{zx}|_{z=0} = 0, \quad \mu_{zy}|_{z=0} = 0, \quad \mu_{zz}|_{z=0} = 0$$

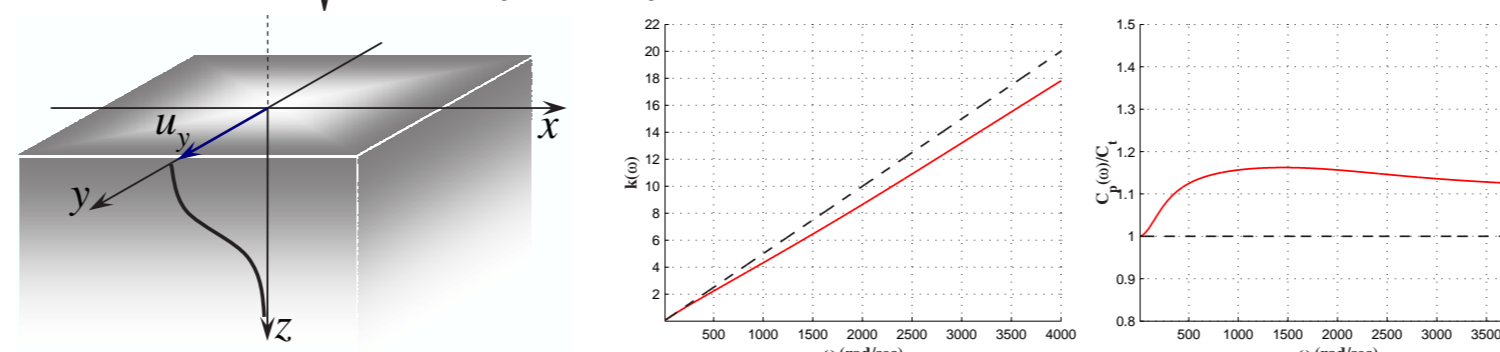
Rayleigh wave with components u_x , u_z and ω_y is determined by the dispersion equation $\det(\mathcal{M}_r(\nu_1, \nu_2, \nu_3)) = 0$, where (*M. Kulesh et al., Journal of Applied Mechanics and Technical Physics, 46(2005), pp. 556-563*).

$$\mathcal{M}_r(p_1, p_2, p_3) = \begin{bmatrix} 2k^2 - \frac{f^2}{C_2^2} & -2ikp_2 & -2ikp_3 \\ 2ikp_1 & 2k^2 - \frac{f^2}{C_3^2} & 2k^2 - \frac{f^2}{C_4^2} \\ 0 & p_2(A_m - \frac{f^2}{C_3^2}) & p_3(A_p - \frac{f^2}{C_4^2}) \end{bmatrix} \\ \nu_1 = \sqrt{k^2 - \frac{f^2}{C_2^2}}, \quad \nu_2 = \sqrt{k^2 - A_m}, \quad \nu_3 = \sqrt{k^2 - A_p}$$



Surface transverse wave with components u_y , ω_x and ω_z has the following dispersion equation: $\det(\mathcal{M}_t(\xi_1, \xi_2, \xi_3)) = 0$, where (*M. Kulesh et al., Acoustical Physics, 52(2006), pp. 186-193*).

$$\mathcal{M}_t(p_1, p_2, p_3) = \begin{bmatrix} \frac{2ik}{1-B} & p_2 \left(2 + \frac{A_m C_3^2 - f^2}{2A^2 C_3^2} \right) & p_3 \left(2 + \frac{A_p C_4^2 - f^2}{2A^2 C_4^2} \right) \\ ikp_1(1+C) & p_2^2 + k^2 C & p_3^2 + k^2 C \\ \left(\frac{C_3^2}{C_4^2} - C - 1 \right) k^2 - p_1^2 \frac{C_3^2}{C_4^2} & ikp_2(1+C) & ikp_3(1+C) \end{bmatrix} \\ \xi_1 = \xi_1 = \sqrt{k^2 - \frac{f^2}{C_2^2} + \frac{4C_3^2}{F C_2^2}}, \quad \xi_2 = \sqrt{k^2 - A_m}, \quad \xi_3 = \sqrt{k^2 - A_p}$$



Note that in the half-space whose dynamic behavior is described by the Cosserat model, in addition to the elliptic surface Rayleigh wave, it is also possible to observe another wave type, a surface wave whose one component is parallel to the boundary surface and perpendicular to the propagation direction as revealed by our solutions to the equations of motion. Both Rayleigh and surface transverse wave have the dispersive character of propagation in the half-space, observation that cannot be explained by the classical elasticity theory.

5 WAVES IN A THIN LAYER

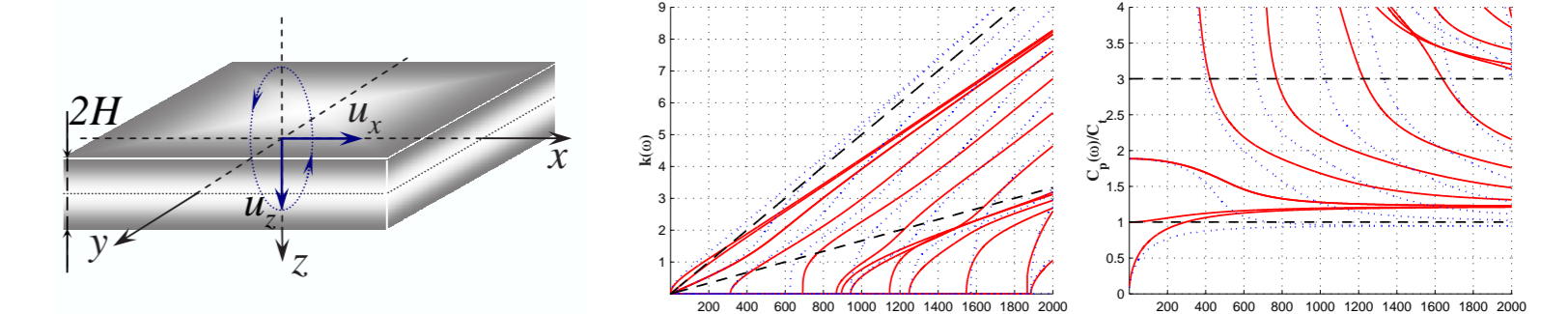
Let us consider a free loaded plate with the thickness $2H$, characteristic length $X_0 = H$

The boundary conditions at the surfaces $z = \pm 1$ require that normal forces and moments are zero and in the dimensionless form are

$$\sigma_{zx}|_{z=\pm 1} = 0, \quad \sigma_{zy}|_{z=\pm 1} = 0, \quad \sigma_{zz}|_{z=\pm 1} = 0 \\ \mu_{zx}|_{z=\pm 1} = 0, \quad \mu_{zy}|_{z=\pm 1} = 0, \quad \mu_{zz}|_{z=\pm 1} = 0$$

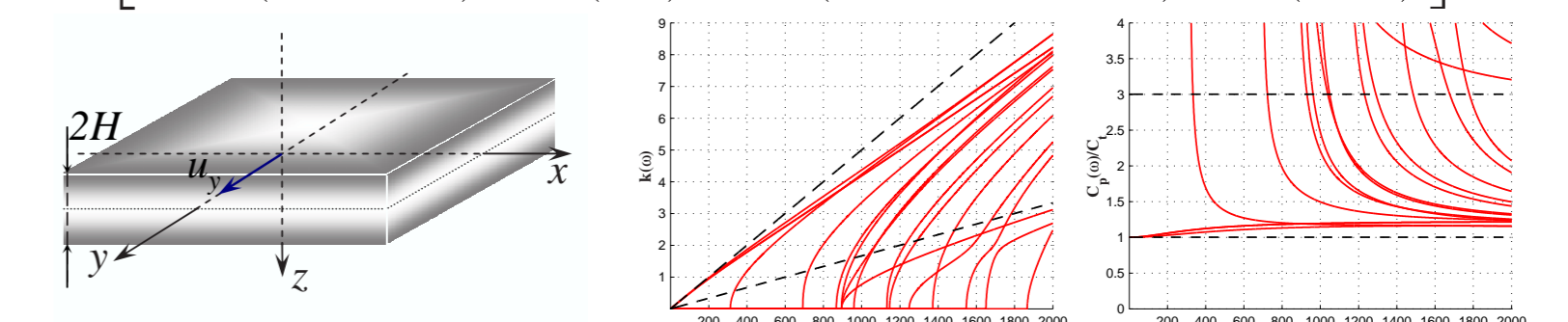
Lamb wave with components u_x , u_z and ω_y is described by the dispersion equation

$$\det \begin{bmatrix} \mathcal{M}_r(\nu_1, \nu_2, \nu_3) \text{diag}(e^{-\nu_n}) & \mathcal{M}_r(-\nu_1, -\nu_2, -\nu_3) \text{diag}(e^{\nu_n}) \\ \mathcal{M}_r(\nu_1, \nu_2, \nu_3) \text{diag}(e^{\nu_n}) & \mathcal{M}_r(-\nu_1, -\nu_2, -\nu_3) \text{diag}(e^{-\nu_n}) \end{bmatrix} = 0$$



Transverse wave with components u_y , ω_x and ω_z has the following dispersion equation:

$$\det \begin{bmatrix} \mathcal{M}_t(\xi_1, \xi_2, \xi_3) \text{diag}(e^{-\xi_n}) & \mathcal{M}_t(-\xi_1, -\xi_2, -\xi_3) \text{diag}(e^{\xi_n}) \\ \mathcal{M}_t(\xi_1, \xi_2, \xi_3) \text{diag}(e^{\xi_n}) & \mathcal{M}_t(-\xi_1, -\xi_2, -\xi_3) \text{diag}(e^{-\xi_n}) \end{bmatrix} = 0$$



In the expressions above $\text{diag}(e^{-p_k})$ is the diagonal matrix.

Note that a qualitatively new wave mode with only one displacement component exists in a free loaded plate within the framework of the Cosserat medium besides well investigated Lamb wave. As in the case of surface transverse wave this new mode also does not have any analogy in the classical elasticity theory.

6 PROBLEM OF NATURAL VIBRATIONS

Solution is found in the form:

$$\mathbf{u} = \boldsymbol{\xi}(\mathbf{x}, \mathbf{y}) e^{-ipt}, \quad \boldsymbol{\omega} = \boldsymbol{\eta}(\mathbf{x}, \mathbf{y}) e^{-ipt}$$

$\boldsymbol{\xi}(\mathbf{x}, \mathbf{y}) e^{-ipt}$, $\boldsymbol{\eta}(\mathbf{x}, \mathbf{y}) e^{-ipt}$ - vectors of natural form of vibrations, p - real frequency of natural vibrations

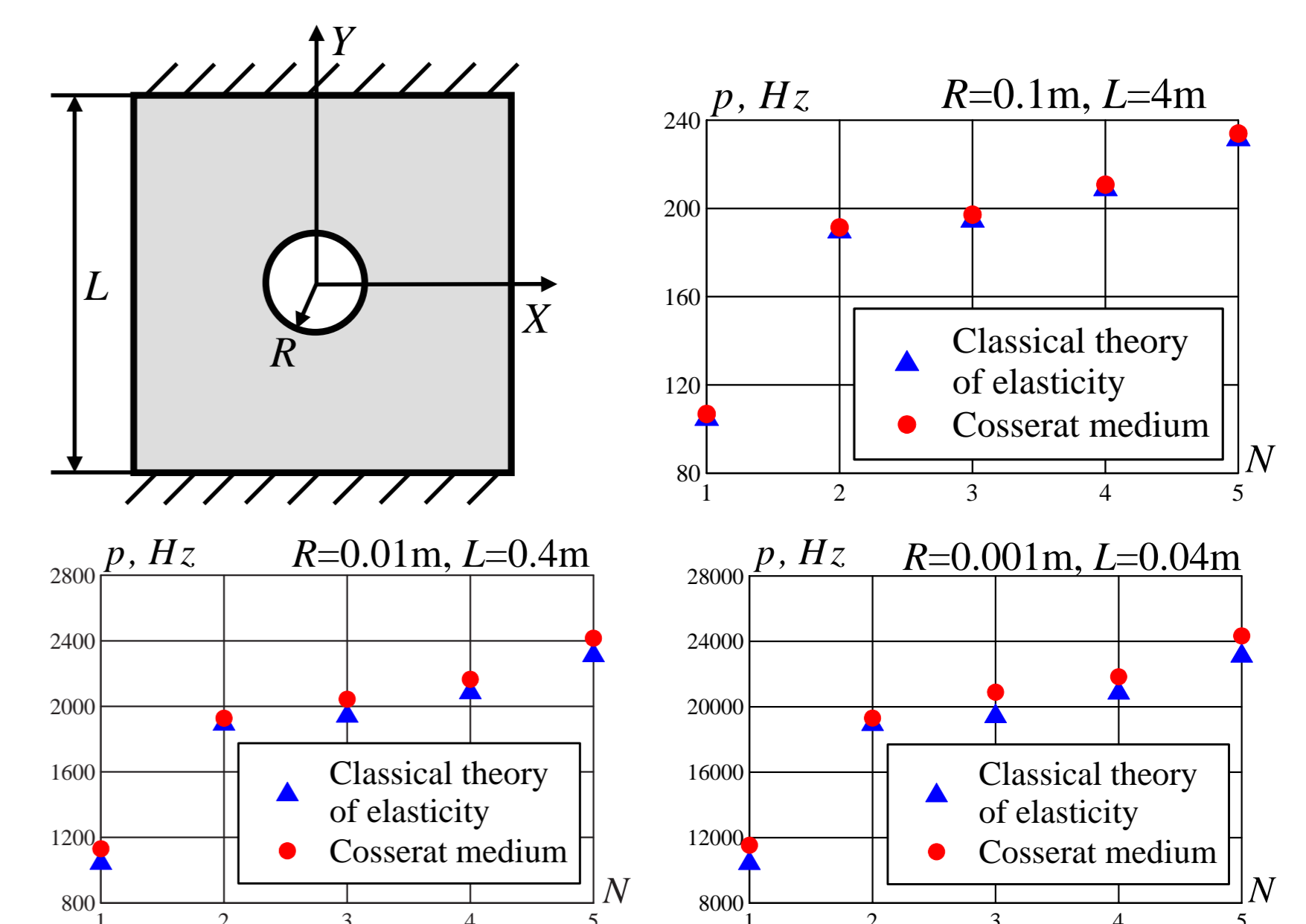
Characteristic equation

$$\det\{[K] - p^2[M]\} = 0$$

[K] - stiffness matrix, [M] - mass matrix

Natural vibrations of the plate with hole

$L = 40R$, N - number of the frequency



CONCLUSION

- This paper is concerned with construction of general motion equation for a linear isotropic Cosserat medium.
- This equation yields a general analytical solution to the linear wave problem, from which one can derive solutions for Rayleigh, Lamb and Stoneley waves and also for a plane volume wave in the Cosserat space.
- A number of marked differences of the obtained solutions from the classical symmetric case have been demonstrated.
- To solve problems on natural vibrations of elastic bodies in the framework of the Cosserat model, The finite-element algorithm has been developed. This algorithm has been applied to some practical problems on natural vibrations.
- The obtained results can be used to identify an important characteristic of the considered system - the spectrum of natural frequencies, which is an experimentally measurable macroparameter responding to the couple-stress properties of the medium.