On the Invariant Index Formulas for Spectral Boundary Value Problems.

Potsdam
August 1998

B.-W. Schulze, B. Sternin, A. Savin

Object of study: index of spectral BVP.

1. Spectral BVP

APS
signature and Dirac operators on mflds with boundary

J. Cheeger
differential-geometry questions on mflds with conical singularities

Naz, Schu, Ste, Sha (1998)
general spectral BVP on mflds with boundary; connections with mflds with singularities

$\boxed{D: C^\infty(M,E) \to C^\infty(M,F)}$ D - elliptic diff. op.
(for short - 1st order)

$$D = -i\frac{\partial}{\partial t} - A\left(t, x, -i\frac{\partial}{\partial x}\right)$$

near boundary

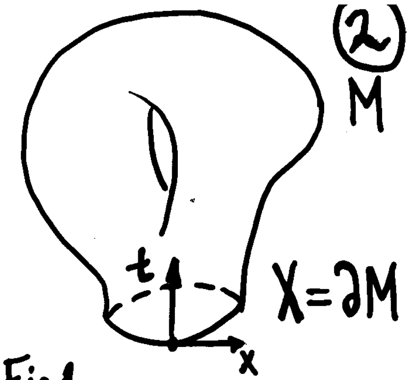

Fig.1

$\boxed{\text{$A(0,x,-i\partial/\partial x)$ tangential operator, has finite number of eigenvalues in angle (see Fig. 2), $\sigma(A)$ has no real eigenvalues for $\xi\neq 0$}}$

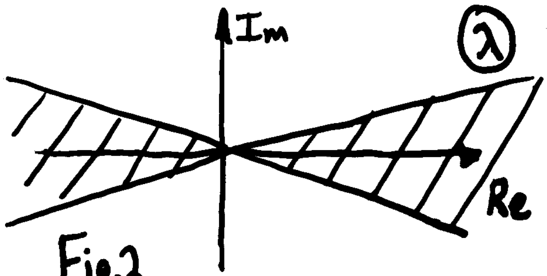

Fig.2

$\boxed{P_+ : C^\infty(\partial M, E) \to C^\infty(\partial M, E)}$ projection onto subspace with $\operatorname{Im}\lambda \le 0$

Spectral BVP for D

$$\boxed{\begin{cases} Du = f \\ P_+(u|_{\partial M}) = g \in \operatorname{Im} P_+ \end{cases}} \quad \text{or} \quad \boxed{\mathcal{D} = \begin{pmatrix} D \\ P_+ \end{pmatrix} : H^s(M,E) \to \begin{matrix} H^{s-1}(M,F) \\ \oplus \\ \operatorname{Im} P_+ \end{matrix}} \quad s > 1/2$$

$\boxed{\text{$\mathcal{D}$ has Fredholm property.}}$

$\boxed{\text{Index is not determined by principal symbol of D.}}$!

Proposition (index invariance). For D_τ, such that

$$\boxed{\text{strip } \{0 < \operatorname{Im}\lambda < \varepsilon\} \text{ has no eigenvalues of } A_\tau} \Longrightarrow \boxed{\operatorname{ind} \mathcal{D}_\tau = \mathrm{Const}.}$$

Theorem (relative index) D_τ - arbitrary elliptic.

$$\boxed{\operatorname{ind} \mathcal{D}_1 - \operatorname{ind} \mathcal{D}_0 = \operatorname{sf}\{A_\tau\}_{\tau=0,1}}$$

2. Statement of Problem

$X = \partial M$

$\mathrm{Ell}(X)$ – principal symbols of diff. op. with no real eigenvalues, $\xi \neq 0$;

$\Sigma \subset \mathrm{Ell}(X)$ – subset;

$\mathrm{Op}(\Sigma)$ – elliptic diff. op. on M

$$-i\frac{\partial}{\partial t} - A(t,x,-i\frac{\partial}{\partial x}), \quad \sigma(A)\Big|_{t=0} \in \Sigma.$$

$\mathrm{Tang}(\Sigma)$ – operators on X with symbols in Σ.

Def.

Class $\mathrm{Op}(\Sigma)$ admits correct decomposition of index formula	$\Longleftrightarrow$	there are $i_f(D)$, $i_b(D)$: 0) $\operatorname{ind} \mathcal{D} = i_f(D) + i_b(D)$; 1) $i_f(D)$ determined by $\sigma(D)$ and constant under deformations of D; 2) $i_b(D)$ determined by tangential A.

Problem: What classes admit correct decompositions?

3. The answer.

Theorem

Class $Op(\Sigma)$ admits correct index decomposition	$\Longleftrightarrow$	for all families $\{A_t\}_{t\in S^1}\subset Tang(\Sigma)$ spectral flow is zero

Corollary 1. Class $Op(\Sigma)$ admits... iff for families of principal symbols $\{\sigma_t\}_{t\in S^1}\subset\Sigma$ number

$$p_!\left([\sigma_t]_{t\in S^1}\right)\in K^1(S^1)\simeq\mathbb{Z}$$

is zero.

$[\sigma_t]_{t\in S^1}\in K^1(S^1\times T^*X)$ – APS difference construction;

$p_!: K^1(B\times T^*X)\to K^1(B)$ (for $B=S^1$) – direct image with parameters B for $p: X\to\{pt\}$

In cohomology

$$\boxed{\left\langle ch\{\sigma_t^+\}_{t\in S^1}\cdot\pi^*Td\,M,[S^1\times S^*X]\right\rangle}$$

$\pi: S^1\times S^*X\to X$,

$\{\sigma_t^+\}_{t\in S^1}\in Vect(S^1\times S^*X)$ generated by subspaces with $\mathrm{Im}\,\lambda<0$

Induces

$$\boxed{sf:\pi_1(Ell(X))\to\mathbb{Z}}\longrightarrow\boxed{sf\in H^1(Ell(X),\mathbb{Q})}.$$

$Op(\Sigma)$ admits correct index formula decomposition	$\Longleftrightarrow$	$i^*sf=0,\quad i:\Sigma\hookrightarrow Ell(X)$

4. (counter) Examples.

a) On M^2 class of all elliptic diff. op. does not admit correct index decomposition.

$$\boxed{\operatorname{Im} P_{t,\varphi} \in \operatorname{Vect}(S^1 \times S^1) \text{ nontrivial} \quad \Big| \quad P_{t,\varphi} \text{ family of proj. } (t,\varphi) \in S^1 \times S^1}$$

$$\Downarrow$$

$$\boxed{\begin{gathered} A_t = (2P_{t,\varphi} - 1)\frac{d}{d\varphi} \quad \text{has} \\ sf\{A_t\}_{t\in S^1} \neq 0 \end{gathered}}$$

b) Hirzebruch operators $\mathcal{D} \otimes 1_E$ with geometrical assumptions
- product structure;
- E with ∇ - flat near ∂M

$$\boxed{\begin{gathered} \operatorname{ind}(\mathcal{D} \otimes 1_E) = i_f + i_b, \\ i_f = n \cdot \operatorname{sign}(M, \partial M) + \int_M L(M)\left(ch(E) - n\right), \\ i_b = -\frac{\eta(-iA \otimes 1_E) - n\eta(-iA) + \dim\ker(A \otimes 1_E)}{2} \end{gathered}}$$

$n = \dim E$