Eta-invariant and index of elliptic operators in subspaces

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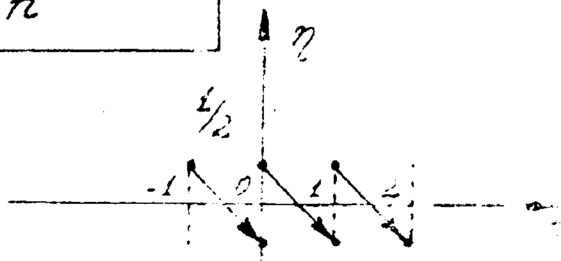
1. η-invariant of Atiyah-Patodi-Singer.

$$\eta(A,s) := \sum_{\lambda_i \neq 0} |\lambda_i|^{-s} \operatorname{sgn} \lambda_i$$

$$\eta(A) := \frac{\dim \ker A + \eta(A,0)}{2}$$

A - elliptic self-adjoint on closed manifold

smooth family A_t → $\eta(A_t)$ piecewise smooth, jumps at points where some eigenvalue changes sign → $\{\eta(A_t)\}$ is smooth



Example.

$$A_t = -i\frac{d}{dx} + t \qquad \lambda_n = n + t$$

on S^1 of length 2π. $\eta(A_t) = \frac{1}{2} - \{t\}$

jumps when some $\lambda_n = n + t$ passes through 0

ord A_t + dim S^1 = 2 (even)

For differential operators, $\operatorname{ord} A + \dim M \equiv 1 \pmod 2$

$\eta(A_t)$ piecewise constant $\Rightarrow$ $\{\eta(A)\}$ - homotopy invariant of principal symbol

Problem. Compute $\{\eta(A)\}$ in terms of $\sigma(A)$

Method: We reduce the computation to a certain index problem

2. η-invariant as a functional of subspaces.

$$A: C^\infty(M,E) \to C^\infty(M,E) \qquad \operatorname{ord} A + \dim M \equiv 1 \pmod 2$$

- self-adjoint elliptic ΨDO

$$\hat{L}_+(A) \subset C^\infty(M,E)$$

- nonnegative spectral subspace

$$\eta(A) = f(L_+(A))$$

- is not a homotopy invariant of the operator;
- is a homotopy invariant of the nonnegative spectral subspace

It is useful to consider general subspaces – images of ΨD projections

1) Subspaces

$$\hat{L} = \operatorname{Im} \hat{P}$$

where

$\hat{P}: C^\infty(M,E) \to C^\infty(M,E)$ - ΨD projection

Symbol of subspaces

$$L = \operatorname{Im} \sigma(P) \subset \pi^* E \in \operatorname{Vect}(S^*M)$$

2) Spectral subspaces

$$\hat{L}_+(A) = \operatorname{Im} \hat{P}_+(A)$$

a) even subspaces

ord A is even:

$$\sigma(A)\cdot\alpha^* = \sigma(A) \Rightarrow \alpha^* L_+(A) = L_+(A)$$

$$\alpha : T^*M \to T^*M \qquad \xi \mapsto -\xi$$

$$\sigma(A)(-\xi) = \sigma(A)(\xi) \Rightarrow$$

$$\Rightarrow L_+(A)(\xi) = L_+(A)(-\xi) \text{ on } S^*M : |\xi| = 1$$

b) odd subspaces

ord A is odd:

$$\sigma(A)\cdot\alpha = -\sigma(A) \Rightarrow L_+(A) \oplus \alpha^* L_+(A) = \pi^* E$$

$$\sigma(A)(-\xi) = -\sigma(A)(\xi) \Rightarrow$$

$$\Rightarrow L_+(A)(\xi) \oplus L_+(A)(-\xi) = E \text{ complements subspaces}$$

Def. $\hat{L}$ is even $\Leftrightarrow L(\xi) = L(-\xi)$ | $\hat{L} \in \widehat{Even}(M)$
$\hat{L}$ is odd $\Leftrightarrow L(\xi) \oplus L(-\xi) = E$ | $\hat{L} \in \widehat{Odd}(M)$

Cases of interest

$\widehat{Even}(M^{odd})$, $\widehat{Odd}(M^{even})$

(when $\{\eta(A)\}$ is homotopy invariant)

<u>Th.</u>

$$\exists !\ d: \widehat{Even}(M^{odd}),\ \widehat{Odd}(M^{ev}) \to \mathbb{R}$$

(invariance) $d(V\hat{L}) = d(\hat{L})$; V is an invertible elliptic operator, $\sigma(V)(x,-\xi) = \sigma(V)(x,\xi)$

(relative index) $L_1 = L_2 \Rightarrow d(\hat{L}_1) - d(\hat{L}_2) = \operatorname{ind}(\hat{P}_2 : \hat{L}_1 \to \hat{L}_2)$
$\hat{P}_2$ is a ΨD projection on $\hat{L}_2$

(complement) $d(\hat{L}) + d(\hat{L}^\perp) = 0$

<u>Additional properties</u>

1) $d(\hat{L} \oplus R) = d(\hat{L}) + \dim R$ if R is finite dimensional
2) $d(C^\infty(M,E)) = 0$
3) $d(\hat{L}) \in \mathbb{Z}[\frac{1}{2}]$

dimensional functional

<u>Th.</u> A is admissible in the sense of Gilkey $\Leftrightarrow$ $\operatorname{ord} A + \dim M \equiv 1 \pmod 2$, $\mathbb{R}_*$-invar. symbol (all terms)

$$\Downarrow$$

$$\eta(A) = d(\hat{L}_+(A))$$

<u>Problem restated</u>: Find $\{d(\hat{L})\}$ — parity condition

Solution will be given via an index formula for subspaces

3. Index formula in subspaces.

$$\mathcal{D} : \hat{L}_1 \to \hat{L}_2 \qquad (*)$$

$\hat{L}_{1,2} \subset C^\infty(M, E_{1,2})$, $(*)$ - restriction of ΨDO

$$\sigma(\mathcal{D}) : L_1 \to L_2$$

- symbol of operator in subspaces (well defined)

$\mathcal{D}$ - elliptic	$\Leftrightarrow$	$\sigma(\mathcal{D})$ - is invertible

Th. $\mathcal{D}$ - elliptic $\Rightarrow$ Fredholm property in Sobolev subspaces

Th. $\operatorname{ind}(\mathcal{D}, \hat{L}_1, \hat{L}_2) = \frac{1}{2} \operatorname{ind} \tilde{\mathcal{D}} + d(\hat{L}_1) - d(\hat{L}_2)$

for $\hat{L}_{1,2} \in \widehat{Even}(M^{odd})$ or $\widehat{Odd}(M^{ev})$

where

$\widehat{Even}$:
$\tilde{\mathcal{D}} : C^\infty(M, E_1) \to C^\infty(M, E_1)$ $\pi^* E_1 = L_1 \oplus L_1^\perp$ $\sigma(\tilde{\mathcal{D}})(\xi) = \sigma^{-1}(\mathcal{D})(-\xi)\,\sigma(\mathcal{D})(\xi) \oplus 1$

$\widehat{Odd}$:
$\tilde{\mathcal{D}} : C^\infty(M, E_1) \to C^\infty(M, E_2)$ $E_i = L_i(\xi) \oplus L_i(-\xi)$ $\sigma(\tilde{\mathcal{D}})(\xi) = \sigma(\mathcal{D})(\xi) \oplus \sigma(\mathcal{D})(-\xi)$

4. Expression for $d(\hat{L})$ via the index in subspaces.

A. A special case.

$$\hat{L}_2 = C^\infty(M,F) \Rightarrow \{d(\hat{L}_1)\} = \{\tfrac{1}{2} \text{ ind } \tilde{D}\}$$

B. The general case.

Th. $\hat{L}$ is admissible (parity condition). Then $[L] \in K(S^*M)/K(M)$ is always a 2-torsion element

⇓

$$\exists N, \sigma : 2^N \hat{L} \xrightarrow{\hat{\sigma}} C^\infty(M,F) \text{ elliptic}$$

From the index formula

$$\{d(\hat{L})\} = \frac{1}{2^N}\left(\text{mod } 2^N\text{-ind}(\hat{\sigma}, 2^N\hat{L}, C^\infty(M,F)) - \frac{1}{2} \text{ mod } 2^{N+1}\text{-ind } \hat{\hat{\sigma}}\right)$$

mod 2^N - ind $(\hat{\sigma}, 2^N\hat{L}, C^\infty(M,F))$ depends only on principal symbols, can be computed with the help of adequate elliptic theory and K-theory

5. Mod n - elliptic theory and mod n - K-theory.

a) mod n - elliptic theory

Operators

$$\mathcal{D}: n\hat{L}_1 \oplus C^\infty(M, E_1) \to n\hat{L}_2 \oplus C^\infty(M, E_2)$$

Trivial operators

1) $g_*: C^\infty(M, E_1) \to C^\infty(M, E_2)$

g is a bundle isomorphism

2) $n\mathcal{D}: n(\hat{L}_1 \oplus C^\infty(M, E_1)) \to n(\hat{L}_2 \oplus C^\infty(M, E_2))$

$Ell(M, \mathbb{Z}_n)$ = the group of classes of stably homotopic operators

b) mod n - K-theory

$$K(Y, \mathbb{Z}_n); \; K(pt, \mathbb{Z}_n) \simeq \mathbb{Z}_n$$

Construction: X a space such that

$$\tilde{K}(X) = \mathbb{Z}_n \qquad (K(X) = \mathbb{Z} \oplus \mathbb{Z}_n)$$

$$K^1(X) = 0$$

$$\boxed{K(Y, \mathbb{Z}_n) \overset{def}{=} K(Y \times X, Y \times pt)}$$

The Moore space

$$X = M_n = \{ D^2 \subset \mathbb{C} \} / \{ e^{i\varphi} \sim e^{i(\varphi + 2\pi k/n)} \}$$

$K(M_n) = \mathbb{Z} \oplus \mathbb{Z}_n$, $K^1(M_n) = 0$

$[\gamma] - 1$ generator of $\mathbb{Z}_n$, γ is a line bundle over M_n with $n\gamma \overset{\sim}{\approx} \mathbb{C}^n$

The Euler characteristic

$$\chi : Ell(M, \mathbb{Z}_n) \longrightarrow K(T^*M, \mathbb{Z}_n)$$

Construction:

Every $[D] \in Ell(M, \mathbb{Z}_n)$ represented by $D : n\hat{L} \to C^\infty(M, F$

We define $\tilde{D}$ an elliptic family on M with parameter space X

$$\sigma(\tilde{D}) : \pi^* F \otimes \mathbb{C} \xrightarrow{\sigma(D)^{-1} \otimes 1} \begin{matrix} nL \otimes \mathbb{C} \\ \| \\ L \otimes \mathbb{C}^n \end{matrix} \xrightarrow{1 \otimes \tau^{-1}} \begin{matrix} nL \otimes \gamma \\ \| \\ L \otimes n\gamma \end{matrix} \xrightarrow{\sigma(D) \otimes 1} \pi^* F \otimes \gamma$$

difference construction	$\chi([D]) \overset{def}{=} [\sigma(\tilde{D})] \in K(T^*M, \mathbb{Z}_n)$

$p : M \times X \to X$ natural projection

$$\begin{array}{ccc} Ell(M, \mathbb{Z}_n) & & \\ \downarrow \chi & \overset{\text{mod}\, n\text{-ind}}{\searrow} & \\ K(T^*M, \mathbb{Z}_n) & \xrightarrow{p_!} & K(pt, \mathbb{Z}_n) \approx \mathbb{Z}_n \end{array}$$

analytical index	$\text{mod}\, n\text{-ind}(D) \overset{def}{=} \text{ind}(D)(\text{mod}\, n) \in \mathbb{Z}_n$

Index theorem

$$\text{mod}\, n\text{-ind}(D) = p_! \chi([D])$$

for $p : M \to \{pt\}$

6. Computation of $d(\hat{L})$.

For the operator $2^N \hat{L} \xrightarrow{\hat{\sigma}} C^\infty(M,F)$,
the index formula gives

$$\boxed{\{d(\hat{L})\} = \frac{1}{2^{N+1}}\, p_!\, [(1 \mp \alpha^*)\sigma]} \quad \text{for } \hat{L} \in \widehat{Even} / \widehat{Odd}$$

$\alpha : T^*M \to T^*M$

$(x,\xi) \mapsto (x,-\xi)$ $\qquad p_! = K(T^*M, \mathbb{Z}_{2^{N+1}}) \to \mathbb{Z}_{2^{N+1}}$

Stabilizing $N \to \infty$

$\mathbb{Z}_{2^N} \subset \mathbb{Z}_{2^{N+1}} \subset \ldots \qquad \varinjlim \mathbb{Z}_{2^N} = \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z}$

$$K(T^*M, \mathbb{Z}_{2^N}) \to K(T^*M, \mathbb{Z}_{2^{N+1}}) \to \ldots \varinjlim K(T^*M, \mathbb{Z}_{2^N}) = K(T^*M, \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z})$$

Th. The element $[L] = [(1 \mp \alpha^*)\sigma] \in K(T^*M, \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z})$ is well defined, and

$\{d(\hat{L})\} = p_!([L]) \in \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z}$

independent of σ

7. Example.

M is a closed oriented Riemannian manifold, odd dimensional

$$A = d\delta - \delta d : C^{\infty}(M, \Lambda^1(M)) \to C^{\infty}(M, \Lambda^1(M))$$

$L_+(A) \subset \pi^* \Lambda^1(M)$ is a one-dimensional even trivial bundle:

$$\sigma : L \longrightarrow \pi^* \mathbb{C} \quad \text{isomorphism}$$

$$\operatorname{ind}(\hat{\sigma}, \hat{L}, C^{\infty}(M)) = \frac{1}{2} \operatorname{ind} \hat{\tilde{\sigma}} + d(\hat{L})$$

$\tilde{\sigma}$ is homotopic to a constant map

$$\Downarrow$$

$$\boxed{d(\hat{L}) = \operatorname{ind}(\hat{\sigma}, \hat{L}, C^{\infty}(M)) \in \mathbb{Z}}$$

Gilkey's theorem on integrality

References:

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