Homotopy classification and Index for elliptic boundary value problems.

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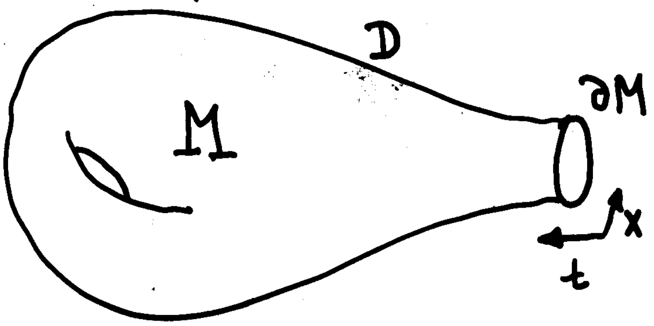
I. Classical boundary value problems.

D - elliptic, $\dim \ker D = \infty$



$$\begin{cases} Du = f \in C^\infty(M,F) \\ B\gamma^{m-1} u = g \in C^\infty(\partial M, G) \end{cases}$$

$u \in C^\infty(M,E)$,
$\operatorname{ord} D = m$.

Ellipticity. $$\begin{pmatrix} D \\ B \end{pmatrix} : H^s(M,E) \longrightarrow \begin{array}{c} H^{s-m}(M,F) \\ \oplus \\ H^{\delta}(\partial M, G). \end{array}$$

$L_+(D) \subset \pi^* E^m$, $\pi : S^*(\partial M) \longrightarrow \partial M$
Cauchy subspace for
$$\sigma(D)\left(x, 0, \xi', -i\frac{d}{dt}\right) u(t) = 0$$
bounded as $t \to +\infty$

Lopatinskii condition

$$\boxed{L_+(D) \xrightarrow{\sigma(B)} \tilde{\pi}^* G \text{ isomorphism on } S^*(\partial M)}$$

Atiyah-Bott obstruction

$$\boxed{\begin{array}{l} 1.\ D \text{ stably admits elliptic boundary value problem;}\\ 2.\ [L_+(D)] \in \tilde\pi^* K(\partial M), \qquad \tilde\pi: S^*(\partial M) \to \partial M;\\ 3.\ \sigma(D)(x,0,\xi',\tau) \sim \sigma'(x) \quad \text{on } \partial M;\\ 4.\ j^*[\sigma(D)] = 0 \in K^1(T^*(\partial M)), \qquad j: T^*M|_{\partial M} \hookrightarrow T^*M. \end{array}}$$

Operators. $D = \sum_{k=0}^{m} D_k(t)\left(-i\frac{\partial}{\partial t}\right)^{m-k}$

Does 3) simplify BVP?

continuous symbols

$D_k(t)$ - pseudodifferential, $\operatorname{ord} D_k(t) = k$

$D_0(t)$ - homomorphism of bundles

$\operatorname{ord} D = 0 \implies D|_{U_{\partial M}} = D_0(t), \quad [\sigma(D)] \in K_c(T^*(M \setminus \partial M))$

$\mathrm{Ell}^0(M) = \{\text{stable homotopy classes of } D,\ \operatorname{ord} D = 0\}$

Th

$$\boxed{\begin{array}{ccc} \mathrm{Ell}^0(M) & \xrightarrow{f} & K_c(T^*(M \setminus \partial M)) \quad \text{isomorphism}\\ D & \longmapsto & [\sigma(D)] \end{array}}$$

Order reduction operators

$$E|_{U_{\partial M}} = E_+ \oplus E_-, \qquad \Lambda_\pm : C^\infty(\partial M, E_\pm) \circlearrowleft, \quad \sigma(\Lambda_\pm) = |\xi'| \cdot 1_{E_\pm}$$

$$\Lambda : C^\infty(M, E) \circlearrowleft, \quad \sigma(\Lambda) = |\xi| \cdot 1_E$$

$$D_\pm = \chi(t)\left[\left(-i\frac{\partial}{\partial t} + i\Lambda_+\right) \oplus \left(+i\frac{\partial}{\partial t} + i\Lambda_-\right)\right] + (1 - \chi(t))\, i\Lambda .$$

$$\boxed{\begin{cases} D_\pm u = f, & u = (u_+, u_-) \\ u_- = g & \in C^\infty(\partial M, E_-|_X) \end{cases}} \quad (*) \quad \text{Elliptic and invertible}$$

<u>Operator D_+: is invertible</u>

$$D_+ = \chi(t)\left(-i\frac{\partial}{\partial t} + i\Lambda_+\right) + (1 - \chi(t))\, i\Lambda \qquad \begin{pmatrix} E_+ = E \\ E_- = 0 \end{pmatrix}$$

$$Ell^m(M) = \left\{ \begin{matrix} \text{stable homotopy classes of } (D, B), \ \operatorname{ord} D = m \\ \text{modulo} \quad (*) \circ D_+^{m-1} \end{matrix} \right\}$$

<u>Th</u> $\boxed{\times D_+^m : Ell^0(M) \longrightarrow Ell^m(M) \quad \text{is isomorphism}}$

<u>Th</u>

1. $\chi \circ (\times D_+^m)^{-1} : Ell^m(M) \longrightarrow K_c(T^*(M \setminus \partial M))$

2. $$\operatorname{ind} \mathcal{D} = p_! \circ \chi \circ (\times D_+^m)^{-1} [\mathcal{D}],$$

$$p : M \setminus \partial M \longrightarrow pt$$

II Boundary value problems for general operators

$$\begin{cases} Du = f \\ Bj^{m-1}u = g \in H^{\delta}(\partial M, G) \end{cases}, \quad L_+(D) \xrightarrow{\sigma(B)} \pi^* G$$

$\dim \operatorname{coker}(D,B) = \infty$ on $S^*(\partial M)$

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$$\boxed{\begin{cases} Du = f \\ Bj^{m-1}u = g \in \operatorname{Im} P \subset H^{\delta}(\partial M, G) \end{cases}} \quad L_+(D) \xrightarrow{\sigma(B)} \operatorname{Im} \sigma(P)$$

$$\mathrm{Ell}^m(M, \partial M) = \{\text{stable homotopy classes } (D,B,P),\ \operatorname{ord} D = m\}$$

Th $\boxed{\times D_+^{m-1} : \operatorname{Spect}(M, \partial M) \longrightarrow \mathrm{Ell}^m(M, \partial M), \quad m \geq 1}$

Spectral BVP:

$D = \frac{\partial}{\partial t} + A$,

P_+ – nonnegative spectral projection for A

$$\begin{cases} Du = f, \\ P_+ u|_{\partial M} = g \in \operatorname{Im} P_+ . \end{cases}$$

Atiyah, Patodi, Singer
Nazaikinskii
Schulze
Sternin
Shatalov

III Homotopy classification of operators with parity conditions.

Sternin
Savin
Schulze Sternin
Savin

$$\boxed{\begin{array}{c}\alpha: T^*X \to T^*X\\ (x,\xi) \mapsto (x,-\xi)\end{array}}$$

$X = \partial M$

Def. $P: C^\infty(X,E) \circlearrowleft$

$$\boxed{\begin{array}{c}\alpha^*\sigma(P) = \sigma(P)\\ \text{Even}\end{array}} \qquad \boxed{\begin{array}{c}\sigma(P)+\alpha^*\sigma(P) = 1_{T^*E}\\ \text{Odd}\end{array}}$$

$$Ell^{ev/odd}(M) = \left\{\begin{array}{c}\text{stably homotopic spectral BVP}\\ P_+ \text{ is even/odd}\end{array}\right\}$$

Th

1. $Ell^{ev/odd}(M^{ev/odd}) \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\chi} K_c(T^*(M \setminus \partial M)) \otimes \mathbb{Z}[\tfrac{1}{2}] \oplus \mathbb{Z}[\tfrac{1}{2}]$ is isomorphism

2. $\operatorname{ind} \mathcal{D} = p_! \, \chi[\mathcal{D}]$, $\quad p: M \setminus \partial M \to pt$

where first term

$$(\mathcal{D}, P_+) \xrightarrow{\text{even}} \left\{\begin{array}{l}\mathcal{D}u = f_1, \ \alpha^*\mathcal{D}v = f_2\\ P_+u|_{\partial M} + (1-P_+)v|_{\partial M} = g\end{array}\right. \qquad \alpha^*\sigma(\mathcal{D})$$

$$(\mathcal{D}, P_+) \xrightarrow{\text{odd}} \left\{\begin{array}{l}\mathcal{D}u = f_1, \ \alpha^*\mathcal{D}^{-1}v = f_2\\ P_+u|_{\partial M} + (1-P_+)v|_{\partial M} = g\end{array}\right. \qquad \alpha^*\sigma(\mathcal{D})^{-1}$$

$$Ell^1(M) \simeq K_c(T^*(M \setminus \partial M))$$

second term

$$(D,P) \longmapsto d(P) \in \mathbb{Z}[\tfrac{1}{2}]$$

$$d: \mathrm{Even}(X^{odd}) \text{ or } \mathrm{Odd}(X^{ev}) \longrightarrow \mathbb{Z}[\tfrac{1}{2}]$$

- relative index: $d(P_1) - d(P_2) = \mathrm{ind}(P_1, P_2)$
- complement $d(P) + d(1-P) = 0$.

Th $\boxed{\begin{array}{l} P = P_+ A, \ A\text{-elliptic, self-adjoint differential} \\ \qquad \mathrm{ord}\, A + \dim X \equiv 1 \pmod 2 \end{array}} \Longrightarrow \boxed{d(\mathrm{Im}\, P) = \eta(A)}$