Homotopy classification and Index of boundary value problems for general elliptic operators. (1)

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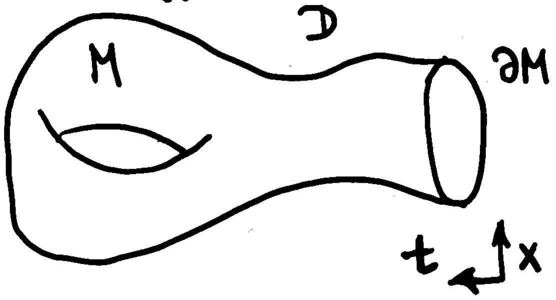
I. Classical boundary value problems.

D - elliptic operator differential



$\dim \ker D = \infty$

$$\boxed{\begin{cases} Du = f, & f \in C^\infty(M,F) \\ B j^{m-1} u = g, & g \in C^\infty(\partial M, G) \end{cases}}$$

$u \in C^\infty(M,E)$, $\operatorname{ord} D = m$,

$j^{m-1} u = \left(u|_{\partial M}, \ldots, \left(-i\frac{\partial}{\partial t}\right)^{m-1} u\Big|_{\partial M} \right)$

B - ψDO on ∂M

Ellipticity

$$\boxed{\begin{array}{l} L_+(D) \subset \pi^* E^m, \ \pi : S^*(\partial M) \to \partial M \\ \text{- Cauchy data of solutions} \\ \sigma(D)\left(x, 0, \xi', -i\frac{d}{dt}\right) u(t) = 0 \\ \text{bounded as } t \to +\infty \end{array}}$$

smooth vector bundle

Def. ($s > m - \frac{1}{2}$)

$$\boxed{\begin{array}{l} \begin{pmatrix} D \\ B \end{pmatrix} : H^s(M,E) \to \begin{array}{c} H^{s-m}(M,F) \\ \oplus \\ H^\delta(\partial M, G) \end{array} \\ \text{- elliptic BVP} \end{array}} \Longleftrightarrow \boxed{\begin{array}{l} \text{Shapiro-Lopatinskij condition} \\ L_+(D) \xrightarrow{\sigma(B)} \pi^* G \\ \text{- isomorphism on } S^*(\partial M) \end{array}}$$

Th $\boxed{(D,B) \text{ - elliptic BVP} \Rightarrow (D,B) \text{ has Fredholm property}}$

Atiyah-Bott obstruction

Following conditions are equivalent:

1. D stably admits elliptic BVP;
2. $[L_+(D)] \in \pi^* K(\partial M)$, $\pi: S^*(\partial M) \to \partial M$;
3. $\sigma(D)(x,0,\xi',\tau) \sim \sigma'(x)$ on ∂M;
4. $j^*[\sigma(D)] = 0 \in K^1(T^*(\partial M))$, $j: T^*M|_{\partial M} \hookrightarrow T^*M$.

Question: can we simplify BVP via 3. ?

class of operators

continuous symbols

$$D = \sum_{k=0}^{m} D_k(t)\left(-i\frac{\partial}{\partial t}\right)^{m-k}$$

$D_k(t)$ - pseudodifferential, $\operatorname{ord} D_k(t) = k$

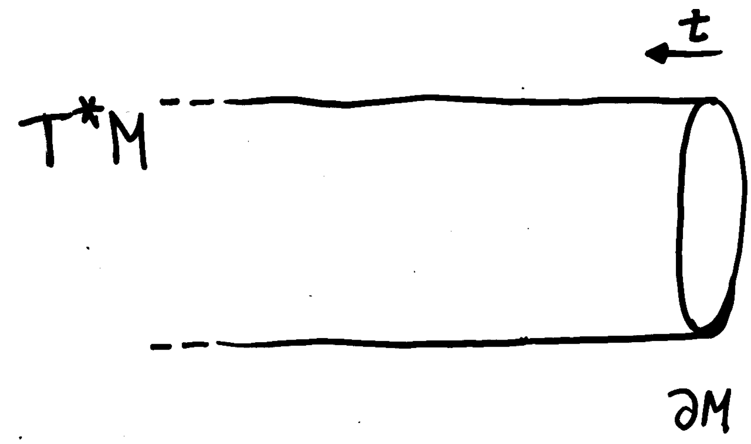
$D_0(t)$ - bundle homomorphism

$\operatorname{ord} D = 0$ $\Rightarrow D|_{U_{\partial M}} = D_0(t)$.

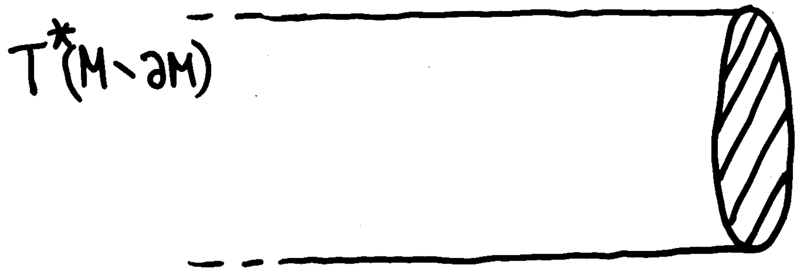
$\boxed{D\text{-elliptic}} \Rightarrow \boxed{D \text{ Fredholm without boundary conditions}}$

$$\mathrm{Ell}^0(M) = \left\{ \begin{array}{l} \text{stable homotopy classes of} \\ \text{elliptic } D,\ \operatorname{ord} D = 0 \end{array} \right\}$$

$$[\sigma(D)] \in K_c(T^*(M \setminus \partial M))$$



$\sigma(D)$ invertible on S^*M



$\operatorname{ord} D = 0$

$\sigma(D)$ invertible on $\partial B^*M = S^*M \cup B^*M\big|_{\partial M}$

<u>Th</u> (homotopy classification)

$$\boxed{\begin{array}{ccl} \mathrm{Ell}^0(M) & \xrightarrow{\chi} & K_c(T^*(M\setminus\partial M)) \text{ - isomorphism} \\ D & \longmapsto & [\sigma(D)] \end{array}}$$

ord $D = m > 0$

Order reduction operators

$E|_{U_{\partial M}} = E_+ \oplus E_-$

ψ - cut-off function

$\psi|_{U_{\partial M}} \equiv 1$

$\Lambda_\pm : C^\infty(\partial M, E_\pm) \circlearrowleft$, $\sigma(\Lambda_\pm) = |\xi'| \cdot 1_{E_\pm}$

$\Lambda : C^\infty(M, E) \circlearrowleft$, $\sigma(\Lambda) = |\xi| \cdot 1_E$

$$D_\pm = \psi(t)\left[\left(-i\frac{\partial}{\partial t} + i\Lambda_+\right) \oplus \left(+i\frac{\partial}{\partial t} + i\Lambda_-\right)\right] + (1-\psi(t))\, i\Lambda .$$

$$\boxed{\begin{cases} D_\pm u = f, & u = (u_+, u_-) \\ u_-|_{\partial M} = g, & g \in C^\infty(\partial M, E_-|_{\partial M}) \end{cases}} \quad (*)$$

elliptic and invertible

operator D_+: $\left.\begin{matrix} E_+ = E \\ E_- = 0 \end{matrix}\right\| \Rightarrow D_+ = \psi(t)\left(-i\frac{\partial}{\partial t} + i\Lambda_+\right) + (1-\psi(t))\, i\Lambda$

$$Ell^m(M) = \left\{ \begin{matrix} \text{stable homotopy classes of elliptic BVP} \\ (D, B), \ \text{ord}\, D = m, \ \text{modulo } (*) \circ D_+^{m-1} \end{matrix} \right\}$$

Th $\boxed{\times D_+^m : Ell^0(M) \longrightarrow Ell^m(M) \quad \text{- isomorphism}}$ the inverse

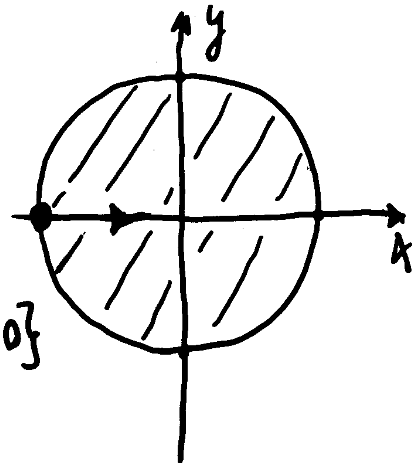
Corollary 1. $\boxed{\chi \circ (\times D_+^m)^{-1} : Ell^m(M) \to K_c(T^*(M \setminus \partial M)) \quad \text{- isomorphism}}$

Corollary 2 (index formula)

$$\boxed{\begin{matrix} \text{ind}\, D = p_! \circ \chi \circ (\times D_+^m)^{-1} [D] \\ p : M \setminus \partial M \to pt, \quad p_! : K_c(T^*(M \setminus \partial M)) \to K(pt) = \mathbb{Z} \end{matrix}}$$

uses AS

II. Boundary value problems for general elliptic operators.



Example: Cauchy-Riemann operator

$$D = \frac{\partial}{\partial \bar z} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \quad z = x+iy$$

$\sigma(D)(\xi,\eta) = -i\xi - \eta \neq 0$ outside $\{\xi=\eta=0\}$

$\boxed{z=-1} \Rightarrow \sigma(D)\left(-i\frac{d}{dx},\eta\right) = \frac{d}{dx} - \eta$

$$\boxed{L_+(D)(-1,\eta) = \begin{cases} 0, & \eta>0 \\ \mathbb{C}, & \eta<0 \end{cases}} \quad \text{on } S^*S^1 = S^1_+ \sqcup S^1_-$$

$L_+(D) \notin \pi^* \mathrm{Vect}(S^1)$, $\pi: S^*S^1 \to S^1$

Atiyah-Bott condition is violated!

Seek well-posed BVP in the form:

$$\begin{cases} Du = f \\ Bj^*u = g, \quad g \in \mathcal{L} \text{ - Banach space} \end{cases}$$

$\ker(D,B) = \ker D \cap \ker Bj^* \simeq j^*\ker D \cap \ker B$

$\operatorname{coker}(D,B) \simeq \operatorname{coker} Bj^*|_{\ker D} \simeq \operatorname{coker} B|_{j^*\ker D}$

reduction to boundary

$\boxed{(D,B,\mathcal{L}) \text{ - well-posed}} \Leftrightarrow \boxed{B: j^*\ker D \to \mathcal{L} \text{ - Fredholm operator}}$

For instance, $\mathcal{L} = j^*\ker D$, $B = \mathrm{id}$.

Hardy space is defined by a ψD projection

$\ker D = \left\{\sum_{k=0}^{+\infty} c_k z^k, c_k \in \mathbb{C}\right\}$, $j^*\ker D \subset H^{s-1/2}(S^1)$

General theory

D elliptic operator violating Atiyah-Bott condition

$$\begin{cases} Du=f \\ Bj^{m-1}u=g, \quad g\in H^{\delta}(\partial M,G) \end{cases} \qquad L_+(D)\xrightarrow{\sigma(B)}\pi^*G \text{ on } S^*(\partial M)$$

$\dim\operatorname{coker}(D,B)=\infty$

boundary values in a subspace || $g\in \operatorname{Im}P\subset H^{\delta}(\partial M,G)$, P - ψD projection

Schultze Sternin Shatalov

$$\boxed{\begin{cases} Du=f \\ Bj^{m-1}u=g, \quad g\in \operatorname{Im}P\subset H^{\delta}(\partial M,G) \end{cases}}$$

boundary symbol

$L_+(D)\xrightarrow{\sigma(B)}\operatorname{Im}\sigma(P)$ on $S^*(\partial M)$

$$\operatorname{Ell}^m(M,\partial M)=\left\{\begin{array}{l}\text{stable homotopy classes}\\ \text{of elliptic BVP}\\ (D,B,P),\ \operatorname{ord}D=m\end{array}\right\}$$

Th (reduction to 1^{st} order operators)

$$\boxed{\times D_+^{m-1}: \operatorname{Spect}(M,\partial M)\longrightarrow \operatorname{Ell}^m(M,\partial M), \quad m\geq 1 \text{ - isomorphism}}$$

Spectral BVP:

$D=\frac{\partial}{\partial t}+A$, A-self-adjoint

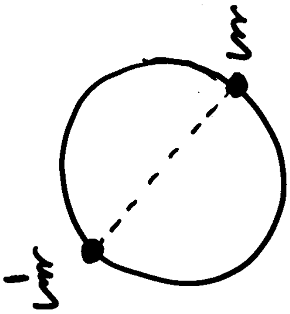
P_+ - nonnegative spectral projection for A

$$\begin{cases} Du=f \\ P_+u|_{\partial M}=g\in\operatorname{Im}P_+ \end{cases}$$

APS NSSS

III. Homotopy classification of BVP with parity conditions.

$$\boxed{\begin{array}{l}\alpha : T^*X \longrightarrow T^*X \\ \qquad (x,\xi) \mapsto (x,-\xi)\end{array}}$$



$X = \partial M$

Def $P : C^\infty(X,E) \circlearrowleft$

$$\boxed{\begin{array}{c}\alpha^*\sigma(P) = \sigma(P) \\ \text{even}\end{array}} \qquad \boxed{\begin{array}{c}\sigma(P) + \alpha^*\sigma(P) = 1_{\pi^*E} \\ \text{odd}\end{array}}$$

Examples

1. $\operatorname{rk} P < \infty \Rightarrow P$ is even.
2. P onto Hardy space $\mathcal{H} \subset L^2(S^1)$ is odd
3. P onto closed forms $\ker d \subset L^2(X, \Lambda^k(X))$ is even

$$Ell^{ev/odd}(M) = \left\{\begin{array}{l}\text{stable homotopy classes of} \\ \text{spectral BVP, } P_+ \text{ - even/odd}\end{array}\right\}$$

<u>Th</u> $$\boxed{\begin{array}{l} Ell^{ev/odd}(M^{ev/odd})\otimes\mathbb{Z}[\tfrac12]\xrightarrow{\chi} K_c(T^*(M\setminus\partial M))\otimes\mathbb{Z}[\tfrac12]\oplus\mathbb{Z}[\tfrac12] \\ \quad - \text{isomorphism}\end{array}}$$

<u>first component of χ (determined by principal symbol)</u>

$$\xrightarrow{\text{even}}\begin{cases} Du=f_1,\ \alpha^*Dv=f_2 \\ P_+u|_{\partial M}+(1-P_+)v|_{\partial M}=g\end{cases} \qquad \alpha^*\sigma(D)$$

(D,P_+) classical BVP

$$\xrightarrow{\text{odd}}\begin{cases} Du=f_1,\ \alpha^*D^{-1}v=f_2 \\ P_+u|_{\partial M}+(1-P_+)v|_{\partial M}=g\end{cases} \qquad \alpha^*\sigma(D)^{-1}$$

$$Ell^1(M)\simeq K_c(T^*(M\setminus\partial M))$$

<u>second component of χ (determined by projection)</u>

$$(D,P)\longmapsto d(\operatorname{Im}P)\in\mathbb{Z}[\tfrac12]$$

dimension functional for subspaces

$$d:\widehat{Even}(X^{odd})\text{ and }\widehat{Odd}(X^{ev})\longrightarrow\mathbb{Z}[\tfrac12]$$

- homotopy invariance;
- $d(\operatorname{Im}P)=\operatorname{rk}P$, if $\operatorname{rk}P<\infty$
- complement $d(\operatorname{Im}P)+d(\operatorname{Im}(1-P))=0$.

<u>Corollary</u> (index formula)

$$\boxed{\begin{array}{l} \operatorname{ind}\mathcal{D}=p_!\chi[\mathcal{D}], \quad p: M\setminus\partial M\to pt \\ \operatorname{ind}(D,P)=\frac12\operatorname{ind}\widetilde{D}-d(\operatorname{Im}P), \\ \widetilde{D}=D\cup\alpha^*D^{\pm1}-\psi DO\text{ on the double }2M\end{array}}$$