Eta-invariant and index of elliptic operators in subspaces

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1. η-invariant of Atiyah-Patodi-Singer.

$$\eta(A,s)=\frac{1}{2}\Big(\sum_{\lambda_i}|\lambda_i|^{-s}\operatorname{sgn}\lambda_i+\dim\ker A\Big)\qquad \operatorname{Re}s\gg 0$$

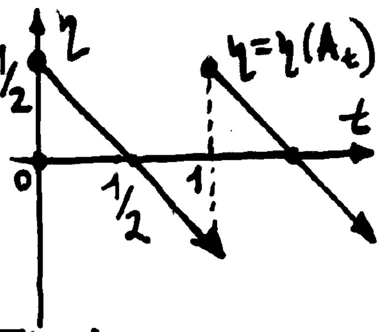
A – elliptic, self-adjoint, $\operatorname{ord}A>0$ on a closed mfld.

Th (APS & Gilkey) $\eta(A)=\eta(A,0)$ well-defined.

Example $A_t=-i\frac{d}{d\varphi}+t$ on S^1, $\lambda_n=n+t$

$\eta(A_t)=\frac{1}{2}-\{t\}$, $t\in\mathbb{Z}+\frac{1}{2}$ – spectral symmetry

$t\in\mathbb{Z}$ – jumps



$\boxed{\text{smooth family } A_t}$ $\Rightarrow$ $\boxed{\eta(A_t)\text{ – piecewise smooth, jumps at points where some eigenvalue changes sign}}$ $\Rightarrow$ $\boxed{\{\eta(A_t)\}\text{ – smooth}}$

P. Gilkey for differential operators. $\boxed{\text{parity condition } \operatorname{ord}A+\dim M\equiv 1\,(\mathrm{mod}\,2)}$ – differential – orders

$\boxed{\eta(A_t)\text{ – piecewise constant}}$ $\boxed{\{\eta(A)\}\text{ – homotopy invariant of principal symbol}}$

Problem $\boxed{\text{Compute } \{\eta(A)\} \text{ in terms of } \sigma(A)}$

Method: Reduce the computation to a certain index problem

2. Subspaces defined by ψD projections

$\eta(A)$ – homotopy invariant, provided no eigenvalue changes sign

Notation $\hat{L}_+(A) \subset C^\infty(M,E)$ – nonnegative spectral subspace

$$\boxed{\eta(A) = f(\hat{L}_+(A)) \text{ – homotopy invariant of subspace}}$$

Def. $\hat{L} \subset C^\infty(M,E)$ – ψD subspace if $\hat{L} = \operatorname{Im} P$, $P : C^\infty(M,E) \circlearrowleft$, $P^2 = P$

$L = \operatorname{Im} \sigma(P) \in \operatorname{Vect}(S^*M)$ – symbol of subspace

$\operatorname{Im} \sigma(P) \subset \pi^* E$

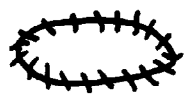
$\pi : S^*M \to M$

symbol of spectral subspace

$$L_+(A) = L_+(\sigma(A)) \in \operatorname{Vect}(S^*M)$$

Examples: 1) $A = -i\dfrac{d}{d\varphi}$, $\hat{L}_+(A) = \left\{ \sum_{k=0}^{+\infty} a_k z^k \right\}$ – Hardy space

$$S^*S^1 = S^1_+ \sqcup S^1_-, \quad L_+(A) = \begin{cases} \mathbb{C}, & \text{on } S^1_+ \\ 0, & \text{on } S^1_- \end{cases}$$



2) $A = \Delta$, $\hat{L}_+(\Delta) = C^\infty(M,E)$

$L_+(\Delta) = \pi^* E.$

3. Subspaces and parity conditions

ord A - even	ord A - odd
$\sigma(A)(-\xi) = \sigma(A)(\xi)$ $L_+(A)(-\xi) = L_+(A)(\xi)$	$\sigma(A)(-\xi) = -\sigma(A)(\xi)$ $L_+(A)(-\xi) \overset{\perp}{\oplus} L_+(A)(\xi) = \pi^* E$

differ. operator

$$\alpha : T^*M \circlearrowleft \quad \alpha(x,\xi) = (x,-\xi)$$

Def Subspace $\hat{L}$ - even $\quad L = \alpha^* L \quad \Big| \quad \widehat{\mathrm{Even}}(M)$

- odd $\quad L \oplus \alpha^* L = \pi^* E \quad \Big| \quad \widehat{\mathrm{Odd}}(M)$

Th $\exists!$ homotopy invariant

$$\boxed{d : \widehat{\mathrm{Even}}(M^{odd}) \text{ and } \widehat{\mathrm{Odd}}(M^{ev}) \longrightarrow \mathbb{R}}$$

(relative dimension) $\dim R < \infty$

$$d(\hat{L} + R) - d(\hat{L}) = \dim R$$

(complement)

$$d(\hat{L}) + d(\hat{L}^{\perp}) = 0.$$

dimension functional

Additional properties

$d(R) = \dim R, \quad d(C^{\infty}(M,E)) = 0.$

$d(\hat{L}) \in \mathbb{Z}[\tfrac{1}{2}]$

A - differential operator, $\operatorname{ord} A + \dim M \equiv 1 \pmod 2 \Rightarrow$

$$\boxed{\eta(A) = d(\hat{L}_+(A))}$$

Problem: restated $\quad \boxed{\text{Find } \{d(\hat{L})\}}$

4. Index of elliptic operators in subspaces

$\hat{L}_{1,2} \subset C^\infty(M, E_{1,2})$, $D: C^\infty(M, E_1) \to C^\infty(M, E_2)$ - ψDO

$D \hat{L}_1 \subset \hat{L}_2$

Def. $\boxed{D: \hat{L}_1 \to \hat{L}_2 \text{ - operator in subspaces}}$ $\boxed{\sigma(D): L_1 \to L_2 \text{ - symbol}}$

Def. $\boxed{D\text{- elliptic}}$ $\Leftrightarrow$ $\sigma(D)$ - isomorphism

Th $\boxed{D\text{- elliptic} \Rightarrow \text{Fredholm operator in the closures of subspaces w.r.t. Sobolev norms}}$

Th (index formula in subspaces)

$$\boxed{\begin{gathered} \operatorname{ind}(D, \hat{L}_1, \hat{L}_2) = \tfrac{1}{2} \operatorname{ind} \widetilde{D} + d(\hat{L}_1) - d(\hat{L}_2) \\ \hat{L}_{1,2} \in \widehat{Even}(M^{odd}) \text{ or } \widehat{Odd}(M^{ev}) \end{gathered}}$$

! index is not determined by principal symbol

$\widehat{Even}$:	$\widehat{Odd}$:
$\widetilde{D}: C^\infty(M, E_1) \to C^\infty(M, E_1)$	$\widetilde{D}: C^\infty(M, E_1) \to C^\infty(M, E_2)$
$\pi^* E_1 = L_1 \oplus L_1^\perp$	$E_i = L_i(\xi) \oplus L_i(-\xi)$
$\sigma(\widetilde{D})(\xi) = \sigma^{-1}(D)(-\xi)\, \sigma(D)(\xi) \oplus 1$	$\sigma(\widetilde{D})(\xi) = \sigma(D)(\xi) \oplus \sigma(D)(-\xi)$

5. Expression of $d(\hat{L})$ in terms of index in subspaces

A. Special case

$$\hat{L}_2 = C^\infty(M,F) \Rightarrow \{d(\hat{L}_1)\} = \{\tfrac{1}{2}\,\mathrm{ind}\,\tilde{D}\}$$

B. General case

$$[L] \in K(S^*M)/K(M) \simeq K^1(T^*M)$$

Th $\boxed{\hat{L} \text{ with parity conditions} \Rightarrow [L] \text{ - 2-torsion element}}$

$$\Downarrow$$

$$\boxed{\exists N, \sigma \quad 2^N \hat{L} \xrightarrow{\hat{\sigma}} C^\infty(M,F) \text{ - elliptic}}$$

From the index formula:

$$\mathrm{ind}\,\hat{\sigma} = \underbrace{\tfrac{1}{2}\,\mathrm{ind}\,\hat{\tilde{\sigma}}}_{\text{usual}} + 2^N d(\hat{L})$$

$$\boxed{\mathrm{mod}\,2^N\text{-}\mathrm{ind}\,(\hat{\sigma}, 2^N\hat{L}, C^\infty(M,F)) \in \mathbb{Z}_{2^N}}$$

determined by principal symbol

6. Modulo n-index computation

$D: n\hat{L} \to C^\infty(M,F)$ | X, $\gamma \in Vect(X)$, $\dim\gamma = 1$,

$\mathrm{mod}\, n\text{-}\mathrm{ind} D \in \mathbb{Z}_n$ | $n([\gamma]-1) = 0 \in \tilde{K}(X)$, $n\gamma \overset{\beta}{\simeq} \mathbb{C}^n$

Elliptic families

$$\mathrm{ind}\, D \otimes 1_{\gamma-1} = (\mathrm{ind} D)([\gamma]-1) \in \tilde{K}(X).$$

$$\mathcal{D} = \quad C^\infty(M,F) \xrightarrow{D^{-1}} n\hat{L} \xrightarrow[\text{isomorphism}]{1\otimes\beta^{-1}} n\hat{L}\otimes\gamma \xrightarrow{D\otimes 1_\gamma} C^\infty(M,F)\otimes\gamma$$

$\mathrm{ind} D([\gamma]-1) = \mathrm{ind}\, \mathcal{D}$, $\quad [\sigma(\mathcal{D})] \in K_c(T^*M \times X)$.

We take $X = M_n$ - *Moore space of $\mathbb{Z}_n$*, $\tilde{K}(M_n) = \mathbb{Z}_n$

$K^1(M_n) = 0$.

Th $\quad \boxed{\mathrm{mod}\, n\text{-}\mathrm{ind} D = p_! [\sigma(\mathcal{D})]}$

$$[\sigma(\mathcal{D})] \in K(T^*M \times M_n, T^*M \times pt) \simeq K(T^*M, \mathbb{Z}_n)$$

$$p_! : K(T^*M, \mathbb{Z}_n) \to K(pt, \mathbb{Z}_n) = \mathbb{Z}_n$$

7. Computation of $\{d(\hat{L})\}$

For operator $2^N \hat{L} \xrightarrow{\hat{\sigma}} C^\infty(M,F)$
the index formula gives

$$\boxed{\{d(\hat{L})\} = \frac{1}{2^{N+1}} p_![(1 \mp \alpha^*)\sigma]} \quad \text{for } \hat{L} \in \widehat{Even}/\widehat{Odd}$$

$\alpha : T^*M \to T^*M$
$(x,\xi) \mapsto (x,-\xi)$ $\qquad p_! : K(T^*M, \mathbb{Z}_{2^{N+1}}) \to \mathbb{Z}_{2^{N+1}}$

stabilization $N \to \infty$

$$\mathbb{Z}_{2^N} \subset \mathbb{Z}_{2^{N+1}} \subset \dots \qquad \subset \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z}$$

$$K(T^*M, \mathbb{Z}_{2^N}) \to K(T^*M, \mathbb{Z}_{2^{N+1}}) \to \dots \qquad \varinjlim K(T^*M, \mathbb{Z}_{2^N}) =$$
$$= K(T^*M, \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z})$$

Th (independent of σ) The element $[L] = [(1 \mp \alpha^*)\sigma] \in K(T^*M, \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z})$ is well-defined, and

$$\{d(\hat{L})\} = p_![L] \in \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z}$$

Exampression via linking form (joint work B. Sternin, A. Savin)

in $\boxed{\{2\,d(\hat{L})\} = [L]\cap([\Lambda]-1)}$

$\Lambda = \Lambda^n(M^n)$ - orientation bundle

linking form in K-theory $\quad \mathrm{Tor}\,K^1(T^*M)\times \mathrm{Tor}\,K^0(M) \xrightarrow{\cap} \mathbb{Q}/\mathbb{Z}$

Example 1 $\boxed{M=\mathbb{R}P^{2n},\ \tilde{K}^0(\mathbb{R}P^{2n})=\mathbb{Z}_{2^n} \quad [\Lambda]-1 \text{ - generator}}$ $\boxed{D = \text{pin}^c\text{-Dirac operator} \quad [D] \text{ generates } K^1(T^*\mathbb{R}P^{2n})}$

$\boxed{\text{nondegeneracy of linking form}} \Rightarrow \{2^n\eta(D)\} = \frac{1}{2}$

Example 2 $\boxed{\exists \text{ differential operators } D^{ev} \text{ on } \mathbb{R}P^{2n}\times S^1 \text{ with } \{\eta(D^{ev})\}\neq 0.}$

Question: Find a geometric operator of even order

Example "Hirzebruch operator" on M^{2n}

Branson 1983
Conn, Sullivan, Teleman 1994

$$\boxed{\begin{array}{l} D = d\delta - \delta d : \Lambda^+\cap\Lambda^n(M) \to \Lambda^-\cap\Lambda^n(M) \\ \Lambda^\pm = \ker(\alpha\mp 1),\quad \alpha = i^{k(k-1)+n}\,* \end{array}}$$

$$\boxed{\begin{array}{l} \ker D = \Lambda^+\cap\ker\Delta\cap\Lambda^n(M) \\ \ker D^* = \Lambda^-\cap\ker\Delta\cap\Lambda^n(M) \end{array}} \Rightarrow \boxed{\mathrm{ind}\,D = \mathrm{sign}\,M}$$

References

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