Index of Elliptic operators on manifolds with covering on boundary – with B. Sternin

1 Motivation

Classical BVP

$\boxed{\text{Atiyah-Bott obstruction to the existence of classical BVP}}$

Most of geometrical operators violate Atiyah-Bott condition

Ways to overcome the obstruction

- BVP in subspaces; (Atiyah-Patodi-Singer problems)
- in the talk → Nonlocal BVP's.

Example:

Dirac operator on cylinder $[0,1]\times X$

$$\not{D} = \frac{\partial}{\partial t} + A$$

A - elliptic self-adjoint operator on X

Well-posed BVP:

$$\begin{cases} \left(\frac{\partial}{\partial t} + A\right) u = f \\ u\big|_{t=0} - u\big|_{t=1} = g, \end{cases}$$

$$g \in C^\infty(X)$$

Manifold M with boundary ∂M

$$\boxed{\text{Boundary } \partial M \text{ is a covering space for a compact manifold } X} \quad : \quad \begin{array}{c} \partial M \\ \downarrow \pi \\ X \end{array}$$

equivalence of function spaces $\Downarrow$

$$\boxed{C^\infty(\partial M) \doteq C^\infty(X,\gamma), \ \gamma \in \mathrm{Vect}(X)} \quad \gamma_x = C^{\pi^{-1}(x)}, x \in X$$
- vector space

Example ($\mathbb{Z}_n$ - manifolds)

$\partial M = X_0 \sqcup X_1 \sqcup \ldots \sqcup X_{n-1}$

$X_0 \overset{g_1}{\simeq} X_1$

$X_0 \overset{g_2}{\simeq} X_2$

$\vdots$

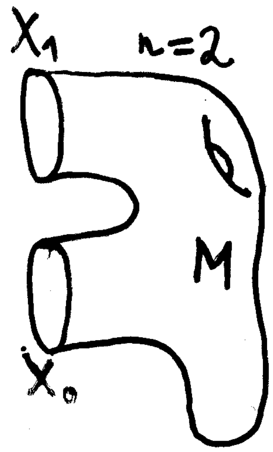
X_0 - manifold (base)

Covering:

$\tilde{\pi} : \partial M \to X_0 = X$

$\pi|_{X_j} = g_j^{-1}$

$C^\infty(\partial M) \simeq C^\infty(X_0, \mathbb{C}^n)$



Def. nonlocal BVP (D,B)

$$\begin{cases} Du = f \\ B\, j_{\partial M} u = g \end{cases} \qquad \begin{array}{l} u \in H^s(M),\ f \in H^{s-m}(M) \\ g \in H^\delta(X,G) \\ j_{\partial M} u = \left(u|_{\partial M}, -i\frac{\partial u}{\partial t}\Big|_{\partial M}, \ldots, \left(-i\frac{\partial}{\partial t}\right)^{m-1} u\Big|_{\partial M} \right) \end{array}$$

D - elliptic operator on M
$\mathrm{ord}\, D = m$

$$B : \bigoplus_{k=0}^{m-1} H^{s-1/2-k}(X,\gamma) \longrightarrow H^\delta(X,G)$$

3) Finiteness theorem

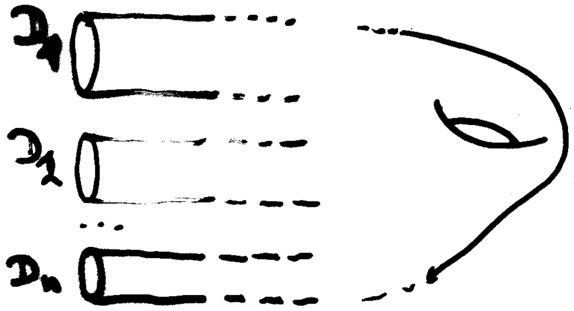
(local) Equivalence with a classical BVP

> near the boundary:
> $C^\infty(\partial M \times [0,1)) \overset{\beta'}{\simeq}$
> $\simeq C^\infty(X \times [0,1), \gamma)$

$$\begin{tikzcd}
C^\infty(\partial M \times [0,1)) \arrow[r, "\begin{pmatrix} D \\ B\beta j \end{pmatrix}"] \arrow[d, equal, "\beta'"'] & C^\infty(\partial M \times [0,1)) \oplus C^\infty(X,G) \arrow[d, equal, "\beta' \oplus 1"] \\
C^\infty(X \times [0,1), \gamma) \arrow[r, dashed, "\begin{pmatrix} \beta' D \beta'^{-1} \\ Bj \end{pmatrix}"] & C^\infty(X \times [0,1), \gamma) \oplus C^\infty(X,G)
\end{tikzcd}$$

classical BVP

does not define global BVP!



Example (trivial covering)

$$\partial M \simeq \underbrace{X \sqcup X \sqcup \ldots \sqcup X}_{n \text{ times}}$$

local BVP: conditions for each of $D_1, D_2, \ldots, D_n$

nonlocal BVP: condition for the direct sum

$$D_1 \oplus D_2 \oplus \ldots \oplus D_n = \begin{pmatrix} D_1 & & 0 \\ & D_2 & \\ 0 & & \ddots \; D_n \end{pmatrix}$$

Def $\boxed{(D,B) \text{ - elliptic boundary value problem}} \iff \boxed{\begin{array}{l} 1)\ D\text{-elliptic on } M \\ 2)\ \text{classical BVP elliptic at } X \times \{0\} \end{array}}$

Th $\boxed{(D,B)\text{-elliptic}} \Rightarrow \boxed{(D,B) \text{ has Fredholm property}}$

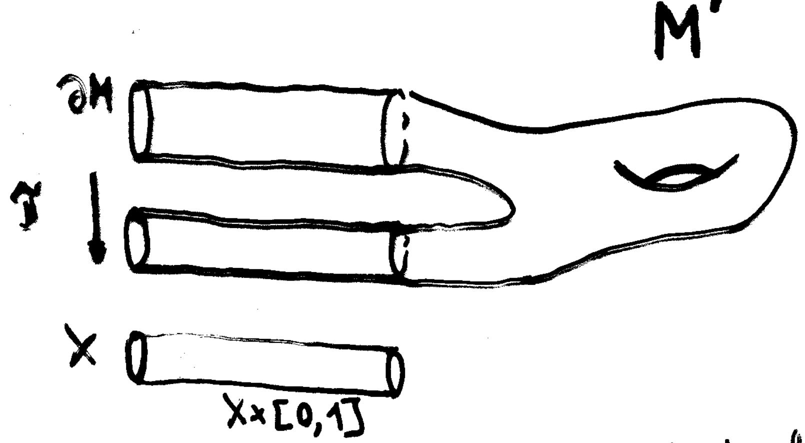
4 Homotopy classification

$$\boxed{\text{An elliptic nonlocal boundary value problem can be reduced to an order zero nonlocal elliptic operator:}}$$

order reduction

$$\mathcal{D}: C^\infty(M) \to C^\infty(M)$$

- on $\partial M \times [0,\varepsilon)$ – vector bundle isomorphism
- on $\partial M \times [0,1)$ equiv. to ΨDO on $X \times [0,1)$
- ΨDO far from the boundary



Def $Ell(M,\pi) = \{$group of stable homotopy classes of nonlocal elliptic operators ord $\mathcal{D}=0\}$

Th (homotopy classification)

$$\boxed{\chi: Ell(M,\pi) \xrightarrow{\simeq} K_0(\mathcal{A}_{T^*M}) \quad \text{- difference construction}}$$

$$\mathcal{A}_{T^*M} \subset C_0(T^*(X\times(0,1)), \operatorname{End}\gamma) \oplus C_0(T^*M')$$

$$\mathcal{A}_{T^*M} = \left\{ u \oplus v \;\middle|\; u|_{t=1} = \beta v|_{\partial M'} \beta^{-1} \right\}$$

Idea:

$$\mathcal{D} \nearrow [\sigma(\mathcal{D})|_{M'}] \in K^0(T^*M')$$

$$\mathcal{D} \searrow [\sigma(\mathcal{D})|_{\partial M \times [0,1]}] \in K^0(T^*(X\times(0,1)))$$

M' and $X \times (0,1]$ cannot be glued along the boundary

Algebras of functions can be "glued" $\Rightarrow$ Single group is defined in C^*-algebra K-theory, since $K^{top}(Y) \simeq K(C_0(Y))$

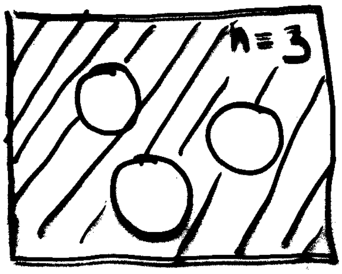
- work with pairs
-

5) Index theorem

Embedding (M,π) in (N,π') if

$$\begin{tikzcd}
f: M \arrow[r, hook] & N \\
\cup & \cup \\
\partial M \arrow[r, hook] \arrow[d, "\pi"'] & \partial N \arrow[d, "\pi'"] \\
X \arrow[r, hook, dashed] & Y
\end{tikzcd}
\quad \longmapsto \quad
\boxed{\begin{array}{l}\text{Direct image map} \\ f!: K(\mathcal{A}_{T^*M}) \to K(\mathcal{A}_{T^*N})\end{array}}$$

Example $M_n = \mathbb{R}^{2d} \setminus \{D_1 \dots D_n\}$



$$\boxed{K(\mathcal{A}_{T^*M_n}) \simeq K(T^*S^{2d-1}) \simeq \mathbb{Z}}$$

Th (index for trivial covering) $\exists\, f: M \hookrightarrow M_n$.

$$\begin{tikzcd}
Ell^0(M,\pi) \arrow[r, "\mathcal{J}"] \arrow[d, "ind"'] & K(\mathcal{A}_{T^*M}) \arrow[d, "f!"] \\
\mathbb{Z} \arrow[r, phantom, "\simeq"] & K(\mathcal{A}_{T^*M_n})
\end{tikzcd}$$

Corollary

$$\operatorname{ind} D = \int_{T^*M'} ch[\sigma(D)]\, Td(T^*M \otimes \mathbb{C}) + \int_{T^*(X\times[0,1])} ch[\tilde{\sigma}(D)]\, Td(T^*X \otimes \mathbb{C})$$

5) some applications

a) involutions on the boundary. (

$\boxed{\begin{array}{l}(M, \partial M) \text{ - compact oriented} \\ \dim M = 4k\end{array}}$

$\boxed{\begin{array}{l}G: \partial M \to \partial M,\ G^2 = Id \\ \text{- orientation-reversing fixed-} \\ \text{point free involution}\end{array}}$

$\Downarrow$

$\boxed{\begin{array}{l}\partial M \\ \downarrow \pi \quad \text{- smooth covering} \\ \partial M/\mathbb{Z}_2\end{array}}$

Hirzebruch operator

$$d + d^* : \Lambda^+(M) \to \Lambda^-(M)$$

near boundary

$$\boxed{\begin{array}{l}\simeq \frac{\partial}{\partial t} + A;\ A: \Lambda^*(\partial M) \to \Lambda^*(\partial M) \\ G^* A = -A G^*\end{array}}$$

$$\boxed{\begin{cases}\left(\frac{\partial}{\partial t} + A\right)\omega = f \\ \frac{(1+G^*)}{2}\, \omega|_{\partial M} = g \in \Lambda^*(\partial M/\mathbb{Z}_2) \simeq \Lambda^*(\partial M)^{\mathbb{Z}_2}\end{cases}}$$

G^*-invariant forms

nonlocal operator

Fredholm property

$\boxed{\text{spec} A: \overset{G^* e}{-\lambda} \quad | \quad \overset{e}{\lambda}}$ $\Rightarrow$ $\boxed{\begin{array}{l}\frac{1+G^*}{2} : \operatorname{Im}\Pi_+(A) \to \Lambda^*(\partial M)^{\mathbb{Z}_2} \\ \text{almost isomorphism}\end{array}}$ $\Rightarrow$ $\boxed{\text{Fredholm property}}$

Th $\boxed{\operatorname{ind}\left(d + d^*, \frac{1+G^*}{2}\right) = \int_M \hat{L}(p_1, \dots, p_k) = \operatorname{sign} M}$

Hsiung 1972

b) Melrose-Freed theory.

<u>for simplicity</u> $\mathcal{D}|_{U_{\partial M}} = \frac{\partial}{\partial t} + A$, A - 1^{st}-order elliptic self-adjoint operator on ∂M

$(\mathcal{D}, \Pi_+)$

Atiyah-Patodi-Singer BVP

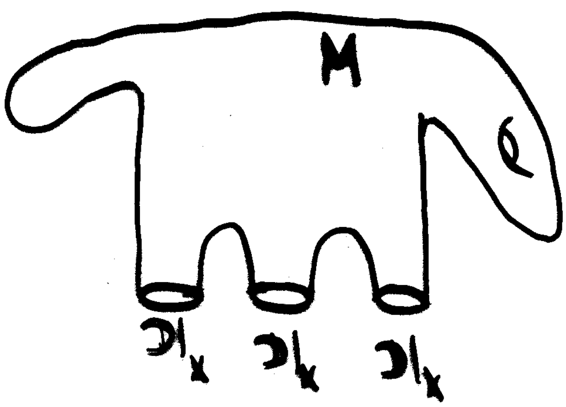
$$\begin{cases} \mathcal{D}u = f, \\ \Pi_+ u|_{\partial M} = g \in \operatorname{Im} \Pi_+ \subset H^{s-1/2}(\partial M) \end{cases}$$

Π_+ - positive spectral projection of A

<u>Assumptions</u>

1) $\partial M = \underbrace{X \sqcup X \sqcup \ldots X}_{n\text{-times}}$

2) $\mathcal{D}$ is the same on all components



$\Downarrow$

$\boxed{\bmod n\text{-}\operatorname{ind}(\mathcal{D}, \Pi_+) \in \mathbb{Z}_n}$ - homotopy invariant of principal symbol $\sigma(\mathcal{D})$

<u>Th</u>

$\bmod n\text{-}\operatorname{ind}(\mathcal{D}, \Pi_+) = \operatorname{ind} \mathfrak{D}$.

$\mathfrak{D}$ - family of nonlocal elliptic BVP parametrized by the Moore space X_n, $\tilde{K}(X_n) = \mathbb{Z}_n$