Boundary value problems on manifolds with fibered boundary

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A. Statement of boundary value problems

- M – manifold with boundary

$\pi : \partial M \xrightarrow{Y} X \qquad x \in X, \quad y \in Y$

Boundary value problem

$$\boxed{\begin{array}{l} Du = f \\ Bu|_{\partial M} = g \in \operatorname{Im} P \subset C^\infty(\partial M, G) \end{array}} \qquad (*)$$

- D – elliptic operator on M;
- P – defined by a family of ψD projections P_x in the fibers;
- B – operator on ∂M

Ellipticity condition for $(*)$?

1) Reduction to boundary

Prop.

(D, B, P) – Fredholm BVP $\iff$ $\hat{L}_+(D) \xrightarrow{B} \operatorname{Im} P$ is Fredholm

$$\hat{L}_+(D) = \operatorname{Im} P_0 \quad \text{– Calderon subspace}$$

2) Ellipticity of operator in subspaces

$\operatorname{Im} \sigma(P_0) \xrightarrow{\sigma(B)} \operatorname{Im} \sigma(P)$ – isomorphism $\iff$ $\operatorname{Im} P_0 \xrightarrow{B} \operatorname{Im} P$ has Fredholm property

P_0 – ψDO on ∂M

$\Downarrow$

$\sigma(P_0)(\xi, \eta)$ smooth for $|\xi| + |\eta| \neq 0$

P – family of ψDO in fibers

$\Downarrow$

$\sigma(P)(\xi, \eta) = \sigma(\eta)$ discontinuous at $\eta = 0$

$\Downarrow$

$\sigma(B)$ disc. as well

Ex.: $f(\xi, \eta) = \operatorname{sgn} \eta$

– This is situation –

Problem

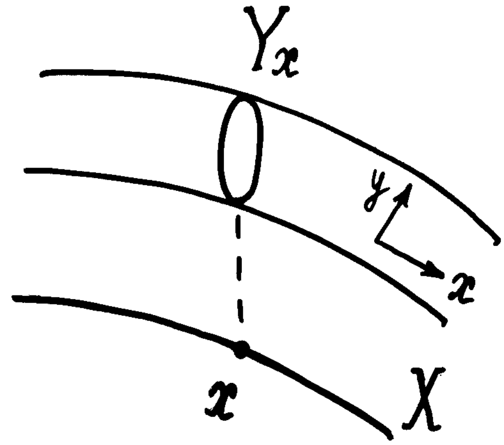
Construct an algebra of operators with disc. symbols on closed manifolds

(Plamenevsky–Rozenblioum – singularities in "physical" variables)

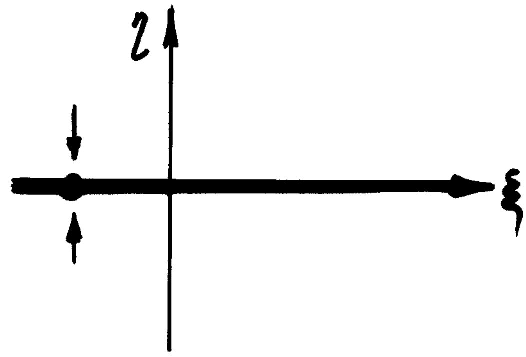
B. Algebra with discontinuous symbols on compact manifolds

Main steps:

- Symbolic algebra;
- Quantization;
- Calculus of operators



$$\pi : M \xrightarrow{Y} X$$



a) Symbolic algebra

Symbols: $\sigma = (a, \hat{a})$

Principal symbol

$a(x, y, \xi, \eta)$ on T^*M

- homog. (order zero) in (ξ, η);
- smooth for $|\eta| \neq 0$;
- has direct. limits as $\eta \to 0$

Operator symbol

$\hat{a}(x, \xi) \in \Psi^0(Y)$ on T^*X

- smooth family for $\xi \neq 0$ of ΨDO of order zero in fibers;
- homogeneous in ξ of order zero

Compatibility condition

$$\sigma(\hat{a}(x, \xi)) = a(x, y, \xi, 0)$$

b) Quantization

$\sigma \longrightarrow \hat{\sigma}$ | explicit formula can be given

c) Calculus

Th.

$$\boxed{\begin{array}{l} \bullet\ \hat{\sigma} \text{ – continuous in Sobolev spaces;} \\ \bullet\ \text{composition formula: } \hat{\sigma}_1\hat{\sigma}_2 = \widehat{\sigma_1\sigma_2} + \text{compact operator} \end{array}}$$

Notation:

$$\boxed{\Psi^0(M,\pi) \text{ – algebra of operators } \hat{\sigma} + \text{compact operator}}$$

Th. (Fredholm property).

$$\boxed{\begin{array}{l}\hat{\sigma} \text{ – Fredholm operator} \\ \quad \text{in Sobolev spaces}\end{array}} \Longleftrightarrow \boxed{\exists\, \sigma^{-1}}$$

C. Application to BVP's

M with boundary, $\pi : \partial M \longrightarrow X$

Boundary value problem:

$$Du = f,$$
$$Bu\big|_{\partial M} = g \in \operatorname{Im} P$$

- P – defined by a family of ΨD projections in the fibers;
- $B \in \Psi^0(\partial M, \pi)$ – operator with disc. symbol

Main Theorem

$$\boxed{\begin{array}{l}(D,B) : H^s(M) \longrightarrow H^{s-1}(M) \oplus \operatorname{Im} P \\ \operatorname{Im} P \subset H^{s-1/2}(\partial M), \quad s > 1/2 \text{ – Fredholm operator}\end{array}}$$

$$\Updownarrow$$

1° Principal symbol
$\operatorname{Im} \sigma(P_0) \xrightarrow{\sigma(B)} \operatorname{Im} \sigma(P)$
is invertible

2° Operator symbol
$\operatorname{Im} \hat{\sigma}(P_0) \xrightarrow{\hat{\sigma}(B)} \operatorname{Im} P$
is invertible

Special case: covering

D. Topological obstruction

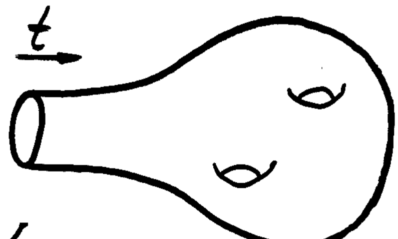
$\boxed{\text{Operator } D \text{ has elliptic boundary value problems}} \Longrightarrow \boxed{\pi_! \left[\sigma(D)\big|_{\partial M} \right] = 0}$

$$\pi_! : K^1(T^*\partial M) \longrightarrow K^1(T^*X)$$

Example 1.

$\boxed{\begin{array}{l} D - \text{geometric operator on } M^{ev} \\ \partial M = X^{ev} \times Y^{odd} \xrightarrow{\pi} X \end{array}} \Longrightarrow \boxed{\text{Obstruction is trivial}}$

$$D\Big|_{U_{\partial M}} \simeq \frac{\partial}{\partial t} + \begin{pmatrix} D_Y & D_X^* \\ D_X & -D_Y \end{pmatrix}$$



D_X, D_Y – geometric operators on X, Y

D_Y – elliptic and self-adjoint with spectral projections $\Pi_\pm = \Pi_\pm\left(y, \frac{\partial}{\partial y}\right)$

$$\begin{cases} \left[\frac{\partial}{\partial t} + \begin{pmatrix} D_Y & D_X^* \\ D_X & -D_Y \end{pmatrix}\right] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\ \Pi_+ u = g_1 \in \operatorname{Im} \Pi_+ \\ \Pi_- v = g_2 \in \operatorname{Im} \Pi_- \end{cases}$$

$\boxed{\text{Elliptic boundary value problem}}$

Example 2

$$\boxed{\partial M = X^{odd} \times Y^{ev} \xrightarrow{\pi} X^{odd}}$$

$$\Downarrow$$

Obstruction:

$$\pi_! \left[\sigma(D)\big|_{\partial M} \right] = \operatorname{ind} D_Y \cdot \left[\sigma(D_X) \right]$$

$\operatorname{ind} D_Y = 0$ ↙ ↘ $\operatorname{ind} D_Y \neq 0$

Elliptic BVP

$$\left[\frac{\partial}{\partial t} + \begin{pmatrix} D_Y & D_X^* \\ D_X & -D_Y \end{pmatrix} \right] \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$u + D_Y^* |D_Y|^{-1} v = g$$

No elliptic BVP

References

A. Savin, B. Sternin. Index defects in the theory of nonlocal boundary value problems and the η-invariant, arXiv: math.KT/0108107