Index defects for boundary value problems and the η-invariant

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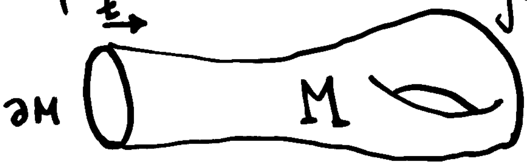
0. Atiyah-Singer index theorem

$\boxed{\text{Elliptic operator } D \text{ on a closed manifold}} \Longrightarrow$

- fredholm operator
- Atiyah-Singer formula
  $ind_a D = ind_t \sigma(D)$

1. Operators on manifolds with boundary



$D: C^\infty(M,E) \to C^\infty(M,F)$
- elliptic operator

$\Longrightarrow$ $\boxed{\text{D requires boundary conditions to obtain fredholm property}}$

For geometric operators: Atiyah-Patodi-Singer boundary value problem

$$(*)\begin{cases} Du = f \\ \Pi_+ u|_{\partial M} = g \in \operatorname{Im}\Pi_+ \end{cases}$$

Π_+ - nonnegative spectral projection of A

$$D|_{U_{\partial M}} \simeq \frac{\partial}{\partial t} + A$$

A - elliptic self-adjoint operator on ∂M
tangential operator

$\boxed{\text{Th (APS 1975) } (*) \text{ has the Fredholm property}}$

Index problem!

$$\boxed{\begin{array}{l} \mathrm{ind}\, D = \dim\ker(D,\Pi_+) - \dim\mathrm{coker}\,(D,\Pi_+) \\ \text{- is not a homotopy invariant of } D \;\; ! \end{array}}$$ (2)

<u>Proposition</u>

$\boxed{D_t \text{ - smooth homotopy}} \Rightarrow \boxed{\mathrm{ind}\, D_0 - \mathrm{ind}\, D_1 = \underbrace{sf\,\{A_t\}_{t\in[0,1]}}}$

spectral flow of the family of tangential operators

<u>Remark</u>. Homotopy invariance is destroyed by tangential operators

<u>Conclusion</u> To obtain a homotopy invariant, index has to be corrected by a functional of A

<u>Statement of index problem</u>

1) find a functional $f(A)$ such that:

$$\underbrace{\mathrm{ind}\, D + f(A)}_{\widetilde{\mathrm{ind}}\, D} \quad \text{is a homotopy invariant of } D$$

2) Compute the invariant topologically

$$\boxed{\widetilde{\mathrm{ind}}\, D = \mathrm{ind}_t D}$$

- $\mathrm{ind}_t D$ - topological index defined by the principal symbol of D;

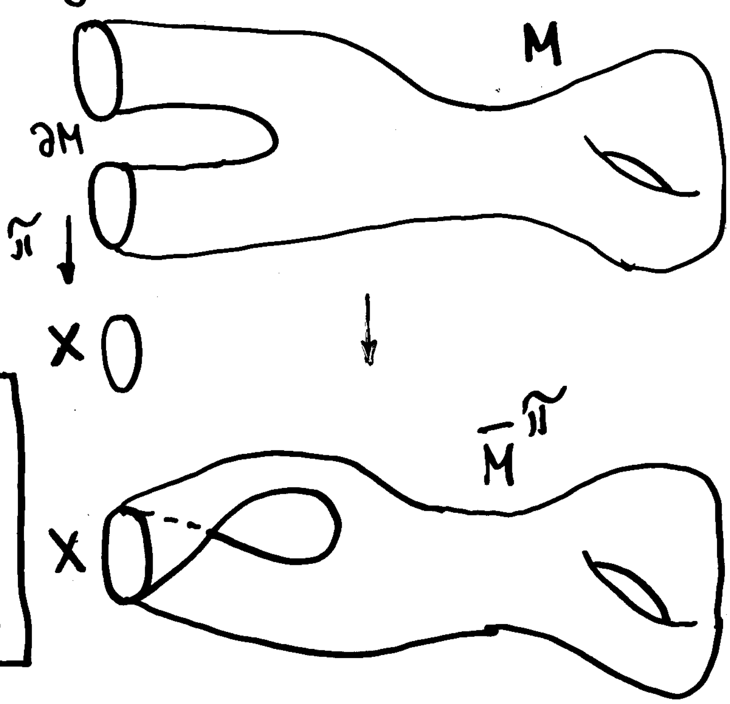
$\Longrightarrow f(A) = \mathrm{ind}_t D - \mathrm{ind}\, D$ - index defect

Manifolds with a covering on the boundary

> M – compact manifold with boundary
> $\partial M \xrightarrow{\tilde\pi} X$ – finite covering with n sheets

$$\Downarrow$$

> singular space $(n \geq 3)$
> $\overline{M}^{\tilde\pi} = M/\sim$
> $x \sim y \Leftrightarrow x = y$ or $x, y \in \partial M$ & $\tilde\pi(x) = \tilde\pi(y)$



Operators on $(M, \mathcal{D})$

(**) $\boxed{\text{Tangential operator lifts from the base}}$ $\qquad A = \tilde\pi^* A_0$

Th $\boxed{\begin{array}{l}\widetilde{\operatorname{ind}} D = \operatorname{ind} D + \eta(A) - n\eta(A_0) \in \mathbb{R}/n\mathbb{Z} \\ \text{– homotopy invariant of } D\end{array}}$

$\boxed{\text{index defect}}$ $\quad \eta(A) - n\eta(A_0)$ – relative η-invariant of Atiyah-Patodi-Singer

$$\eta(A) = \frac{1}{2}\left(\sum_i \frac{\operatorname{sgn}\lambda_i}{|\lambda_i|^s} + \dim\ker A\right)\Bigg|_{s=0}.$$

Topological index construction

On closed smooth manifold (topological K-theory)

$$\boxed{\begin{array}{l}\text{Poincare duality:}\\ \langle , \rangle : K^0_c(T^*M)\times K^0(M)\to \mathbb{Z}\end{array}} \qquad \boxed{\begin{array}{l}\text{topological index}\\ \mathrm{ind}_t D = \langle [\sigma(D)], [1]\rangle\end{array}}$$

$[\sigma(D)]\in K^0_c(T^*M)$

$1\in \mathrm{Vect}(M)$

our case: $\boxed{D \text{ satisfies } (**)} \Rightarrow [\sigma(D)]\in K^0_c(\overline{T^*M}^{\pi})$

Singular manifolds (K-theory of algebras)

Noncommutative geometry

Th $\boxed{\begin{array}{l}\text{There exists a Poincare duality}\\ \langle , \rangle : K^0_c(\overline{T^*M}^{\pi})\times K_0(\mathcal{A}_{M,\pi})\to \mathbb{Z}\end{array}}$

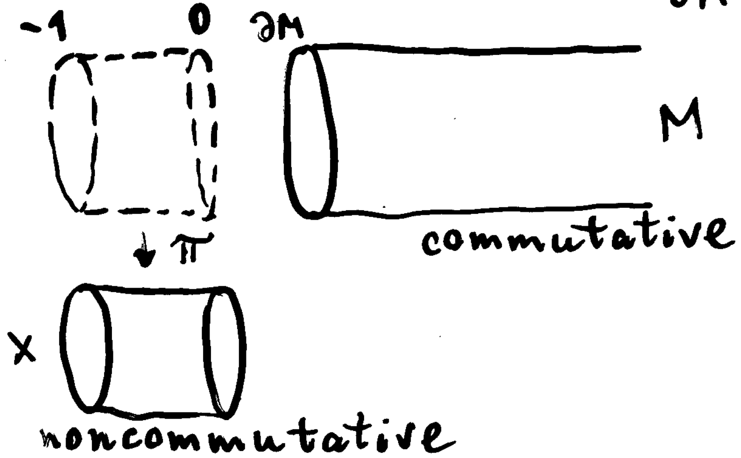
C^*-algebra $\mathcal{A}_{M,\pi}$

1) flat bundle $\pi_! 1 \in \mathrm{Vect}(X)$

$(\pi_! 1)_x = C^\infty(\pi^{-1}(x))$ $\Big\}$ $C^\infty(\partial M)\overset{\beta}{\simeq} C^\infty(X, \pi_! 1)$

2) $\mathcal{A}_{M,\pi}\subset C_0\big((-1,0]\times X, \mathrm{End}\,\pi_! 1\big)\oplus C(M)$

$$= \{ u\oplus v \mid u|_{t=0} = \beta\, v|_{\partial M}\,\beta^{-1}\}$$



Proposition

$\boxed{\text{Pair } M, \pi} \Longrightarrow \boxed{\text{class } \gamma\in K_0(\mathcal{A}_{M,\pi}, \mathbb{Q}/n\mathbb{Z})}$

Index defect formula

Th For a regular covering $\mathfrak{X}$:
$$\boxed{\widetilde{\operatorname{ind}} D = \langle [\sigma(D)], \gamma \rangle \quad \in \mathbb{Q}/n\mathbb{Z}}$$
$$\widetilde{\operatorname{ind}} D = \operatorname{ind} D + \eta(A) - n\eta(A_0)$$

Corollaries

1) trivial covering $\Longrightarrow$ mod n – index formula of Freed-Melrose

Higson 90
Zhang 96
Botvinnik 99
Rosenberg 01

2) taking fractional part $\Longrightarrow$ APS "index" formula for flat bundles

3) computation of the η-invariant

$D = D\!\!\!/_M$ – Dirac operator
Assume spin structure, metric are pull-backs near boundary

$$\{\eta(D\!\!\!/_X)\} = \frac{1}{n}\left[\int_M \hat{A}(M) - \langle [\sigma(D)], \gamma \rangle\right]$$

Not discussed here:

- Poincaré isomorphisms & K-homology;
- Lefschetz formula computation of $\widetilde{\operatorname{ind}}$;