Eta invariant and parity conditions

(example of a nontrivial η-invariant for a second order operator)

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I. Historical introduction

a) Definition of the η-invariant

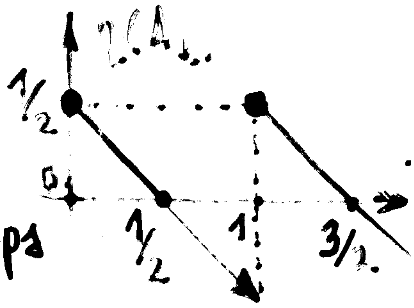
$$\eta(A,s)=\frac{1}{2}\Big(\sum_{\lambda_i} \operatorname{sgn}\lambda_i\, |\lambda_i|^{-s} + \dim\ker A\Big) \qquad \operatorname{Re} s \gg 0$$

A – elliptic, self-adjoint operator on a closed mfld M

Th (APS & Gilkey) $\eta(A)=\eta(A,0)$ – well-defined

Example 1 $A_t=-i\frac{d}{d\varphi}+t$ on S^1, $\lambda_n=n+t$

$\eta(A_t)=\frac{1}{2}-\{t\}$ $\quad t\in\mathbb{Z}$ - jumps



smooth family	piecewise smooth	smooth
A_t	$\eta(A_t)$	$\{\eta(A_t)\}\in\mathbb{R}/\mathbb{Z}$

jumps at points where eigenvalue changes its sign

We consider $\{\eta(A)\}$ to eliminate jumps

b) Situations when $\{\eta(A)\}$ is homotopy invariant

(i) (APS75) Operators with coefficients in flat bundles

(ii) (Gilkey89) Differential operators with parity condition

Parity conditions

Th (Gilkey 89)

$$\boxed{\begin{array}{c}\text{differential operator } A \\ \operatorname{ord} A + \dim M \equiv 1 \pmod 2\end{array}} \Rightarrow \boxed{\{\eta(A)\} \text{ - homotopy invariant}}$$

parity condition

C) On the nontriviality of $\{\eta(A)\}$

(i) Atiyah-Patodi-Singer: see Example 1.

(ii) operators with parity conditions

- even-dimensional manifolds (Gilkey 85)

$$\boxed{\begin{array}{l}M^{ev}\text{-orientable} \\ \not{D} \text{ - Dirac operator}\end{array}} \Rightarrow \boxed{\begin{array}{c}\operatorname{spec} \not{D} \text{ - symmetric } \lambda \leftrightarrow -\lambda \\ \{\eta(\not{D})\} = \{\tfrac{1}{2} \dim\ker \not{D}\}\end{array}}$$

Th (Gilkey 85)

$$\boxed{\{\eta(\not{D}_{\mathbb{R}P^{2n}})\} = \frac{1}{2^{n+1}}}$$

$\not{D}_{\mathbb{R}P^{2n}}$ - pin^c Dirac operator on nonorientable $\mathbb{R}P^{2n}$

Applications to pin^c-bord
Bahri-Gilkey 8

(A) - odd-dimensional manifolds (?)
even order operators

We solve problem A in this talk

II. Solution of Problem A

a Construction of the operator and the main result

Manifold: $\mathbb{R}P^{2n} \times S^1$

Operator: $A_n =$

$$
= \begin{cases}
\begin{pmatrix} 2\sin\varphi\left(-i\frac{\partial}{\partial\varphi}\right)\not{D} - i\cos\varphi\not{D} & \Delta_x e^{-i\varphi} + \left(-i\frac{\partial}{\partial\varphi}\right)e^{i\varphi}\left(-i\frac{\partial}{\partial\varphi}\right) \\ \Delta_x e^{i\varphi} + \left(-i\frac{\partial}{\partial\varphi}\right)e^{-i\varphi}\left(-i\frac{\partial}{\partial\varphi}\right) & 2\sin\varphi\left(i\frac{\partial}{\partial\varphi}\right)\not{D} + i\cos\varphi\not{D} \end{pmatrix}, & \varphi \in (0, \\
\begin{pmatrix} 0 & e^{-i\varphi}\Delta \\ \Delta e^{i\varphi} & 0 \end{pmatrix}, & \varphi \in (\pi, 2\pi)
\end{cases}
$$

Where $\not{D}$ - pin^c Dirac operator on $\mathbb{R}P^{2n}$

$$\Delta_x = \not{D}^2$$

$$\Delta = \not{D}^2 + \left(-i\frac{\partial}{\partial\varphi}\right)^2$$

$$A_n^2 \equiv \Delta^2$$ (modulo lower order terms)

Th

$$\boxed{\{2\eta(A_n)\} = \frac{1}{2^{n-1}}} \qquad (*)$$

B Main theorem

Th A-differential operator : $\operatorname{ord} A + \dim M \equiv 1 \pmod 2$

(ScSS

Then

$$\boxed{\begin{array}{l} \{2\eta(A)\} = \langle [\sigma(A)], [1-\Lambda^n(M)] \rangle, \\ \text{where } [\sigma(A)] \in K^1_c(T^*M),\ \Lambda^n(M) \text{ - orientation} \\ \text{bundle, } \langle , \rangle \text{ - Poincare duality:} \\ \operatorname{Tor} K^1_c(T^*M) \times \operatorname{Tor} K^0(M) \longrightarrow \mathbb{Q}/\mathbb{Z} \end{array}}$$

Proof (sketch) For even order operators:

a)
$$\boxed{\begin{array}{ccc} Ell^{ev}(M) & \xrightarrow{\approx} & K(P^*M) \\ A & \longmapsto & \operatorname{Im} \Pi_+ \sigma(A) \end{array}} \in Vect(S^*M)$$

where $Ell^{ev}(M)$ - group of homotopy classes for even order operators;

$P^*M = S^*M/\mathbb{Z}_2$ - projectivization.

(Gil

b) $\boxed{K(P^*M)/K(M) \text{ - 2-torsion group if } \dim M \text{ odd}}$

From a) and b) we obtain:

For any operator A $\exists N$:

$$2^N A \overset{B_t}{\sim} \Delta_{E'} \oplus - \Delta_{E''}$$

operator

$$A = \sum_{|\alpha| \le m} a_\alpha(x)(-1)^{|\alpha|} \frac{\partial^\alpha}{\partial x^\alpha}$$

$i = 1, \ldots, n$

$x_1 \ldots x_n$

$$\frac{\partial^\alpha}{\partial x^\alpha} \overset{def}{=} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

principal symbol

$$\sigma(A)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$$

$$\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

globally defined for $(x, \xi) \in T^*M$

c) Lemma (Gilkey)

$$\eta(B_1) - \eta(B_0) = sf(B_t)_{t\in[0,1]}$$

d)

$$B_1 = \Delta_{E'} \oplus -\Delta_{E''}$$
$$B_0 = 2^N A$$
$$\Big\Downarrow \quad \eta(\Delta) = 0$$
$$\{\eta(A)\} = -\frac{1}{2^N} \bmod 2^N\text{-}sf(B_t) \in \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z}$$

APS

e) Th

mod m-index theorem

$$\boxed{\begin{array}{l}\bmod m\text{-}sf(B_t) = -p_![\sigma(B_t)], \\ \text{where } [\sigma(B_t)] \in K_c(T^*M, \mathbb{Z}_m),\ p_! : K_c(T^*M,\mathbb{Z}_m) \to \mathbb{Z}_m\end{array}}$$

APS, Freed, Melrose Zhang

f) $$\{2\eta(A)\} = \frac{1}{2^N} p_!\big((1+\alpha^*)[\sigma(B_t)]\big)$$

where $\alpha : T^*M \longrightarrow T^*M$

$$(x,\xi) \longmapsto (x,-\xi)$$

$$\alpha^* : K_c(T^*M, \mathbb{Z}_{2^N}) \longrightarrow K_c(T^*M, \mathbb{Z}_{2^N})$$

g) $\boxed{\alpha^* = (-1)^{\dim M}[\Lambda^n(M)]}$ (based on work of Karoubi)

$$\{2\eta(A)\} = \frac{1}{2^N} p_!\big([1-\Lambda^n(M)]\,[\sigma(B_t)]\big)$$
$$\stackrel{def}{=} \langle [\sigma(A)], [1-\Lambda^n(M)] \rangle .$$

$\sqcap$

Formula can be written in the Atiyah-Singer form

$\boxed{\Lambda^n(M^n) \text{ - } \mathbb{Z}_2\text{-bundle}}$

$\Lambda^n(M^n) \simeq f^*\gamma, \quad f: M^n \to B\mathbb{Z}_2 = \mathbb{R}P^\infty$

γ - tautological line bundle

<u>Th</u>

$$\boxed{\{2\eta(A)\} = f_![\sigma(A)]}$$

Atiyah-Singer formula:

$$\operatorname{ind} D = f_![\sigma(D)]$$

$$f_!: K_c(T^*M) \to K(pt) = \mathbb{Z}$$

$$f_!: K_c^1(T^*M) \to K_c^1(T^*\mathbb{R}P^\infty) = \mathbb{Z}_{2^\infty} = \mathbb{Z}[\tfrac{1}{2}]/\mathbb{Z}$$

<u>Corollaries</u>

1) $\eta(A) \in \mathbb{Z}[\frac{1}{2}]$

2) M - orientable $\Rightarrow \eta(A) \in \frac{\mathbb{Z}}{2}$

M^{2k+1} - nonorientable $\Rightarrow 2^{k+1}\eta(A) \in \mathbb{Z}$.
M^{2k}

C The proof of formula (*)

$$\{2\eta(A_n)\} = \langle [\sigma(A_n)], [1 - \Lambda^{2n+1}(\mathbb{R}P^{2n} \times S^1)] \rangle$$

a) Lemma $[\sigma(A_n)] = [\sigma(\mathcal{D})] \cdot [\sigma(D_0)]$,

where D_0 — an elliptic ψDO on S^1, $\operatorname{ind} D_0 = 2$

b) $$\{2\eta(A_n)\} = \langle [\sigma(\mathcal{D})][\sigma(D_0)], [1 - \Lambda^{2n}(\mathbb{R}P^{2n})] \rangle =$$

$$= \operatorname{ind} D_0 \langle [\sigma(\mathcal{D})], [1 - \Lambda^{2n}(\mathbb{R}P^{2n})] \rangle =$$

$$= 2 \cdot \{2\eta(\mathcal{D})\} = 2 \cdot \frac{1}{2^n} = \frac{1}{2^{n-1}}.$$

use Gilkey's result

A geometric construction of second order operators

First order operator D	Second order operator D
$\Downarrow$	$\Downarrow$
Principal symbol $\sigma(D)(x,\xi)$ – linear in ξ	Principal symbol $\sigma(x,\xi)$ quadratic in ξ

Idea: use quadratic transformation (Gilkey 1981) of covariables ξ

$$\boxed{\text{linear symbol}} \longrightarrow \boxed{\text{quadratic symbol}}$$

Geometry

$X \subset M^m$, trivial normal bundle

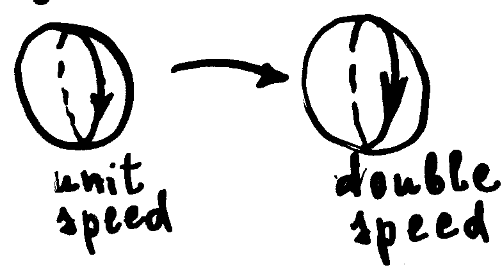
$U_X \simeq D^n \times X$

$(n=1)$

Quadratic transformation

$$\boxed{\begin{aligned} h : S^{m-1} &\longrightarrow S^{m-1} \\ (\eta, \tau) &\longmapsto (2\tau\eta, \tau^2 - \eta^2) \end{aligned}}$$

degree 2 map

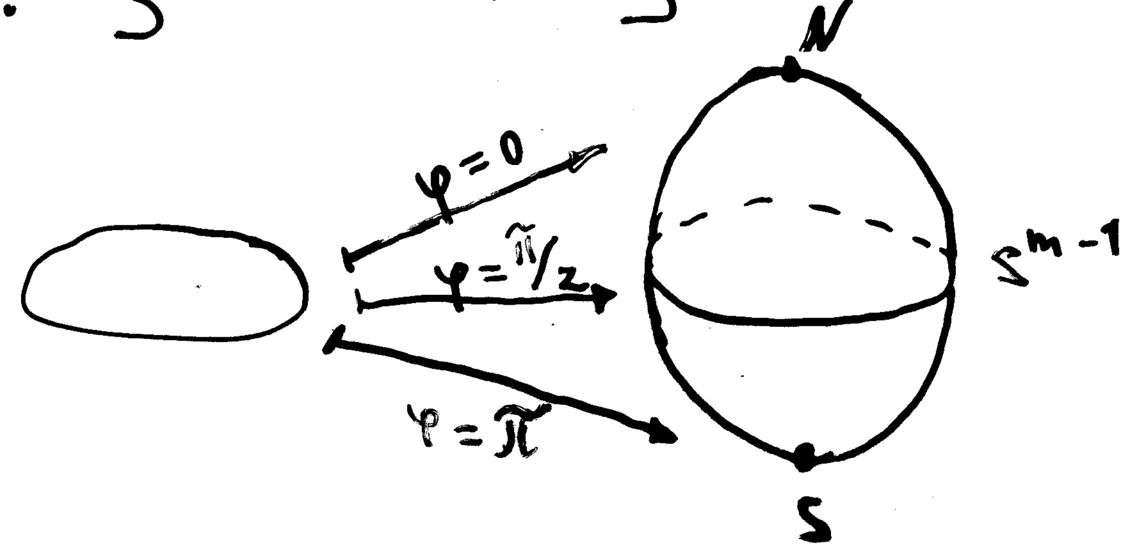


Deformation

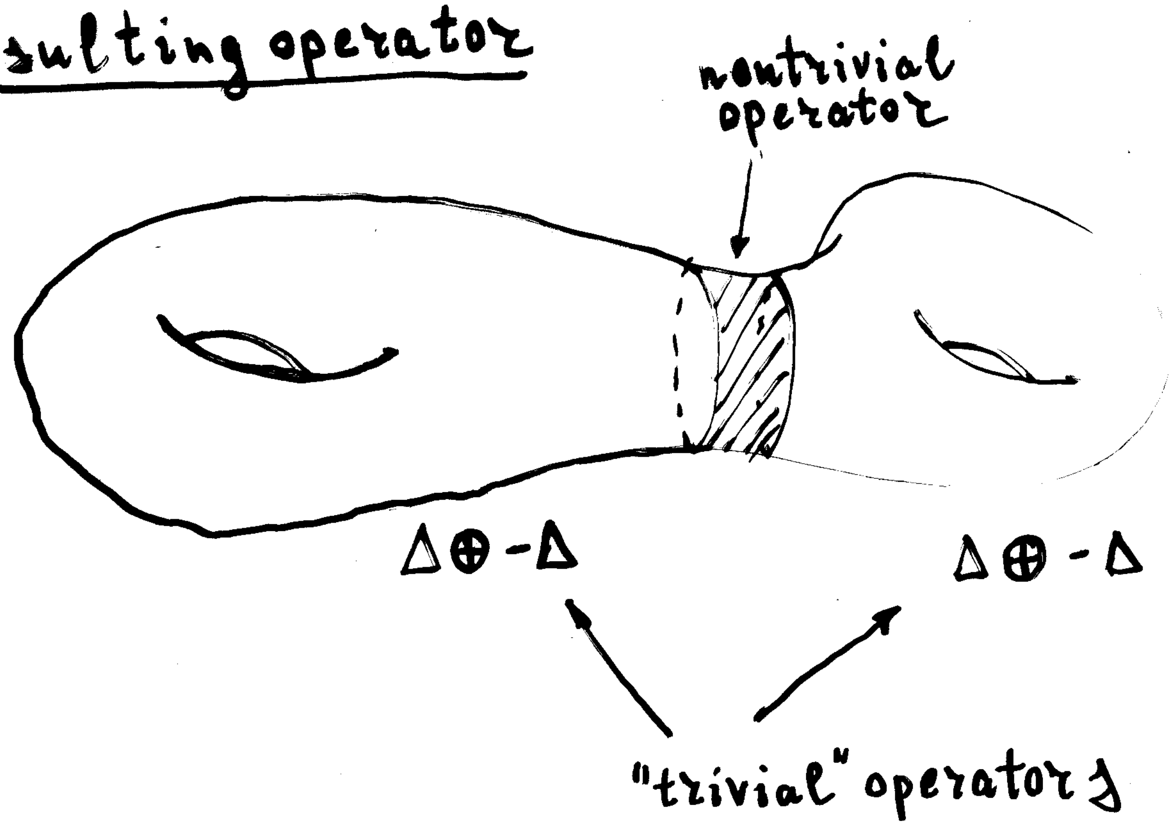
$$\boxed{\begin{aligned} \Phi : [0,\pi] \times S^{m-1} &\longrightarrow S^m \\ (\varphi, \xi) &\longmapsto (\cos\varphi\, \xi^2, \sin\varphi\, h(\xi)) \end{aligned}}$$

linear symbol on X:		quadratic symbol on U_X
$\sigma(\tau, \xi)$	$\Longrightarrow$	$\sigma \circ \Phi$

$$\Phi(\varphi,\cdot): S^{m-1} \longrightarrow S^m$$



Resulting operator



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